EXTREMAL LENGTH ON CURVES AND GRAPHS

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In this short note we will relate a notion "extremal length" on elastic graphs and to extremal length on surfaces. This is part of a larger work on rational maps. The elastic graphs Γ and their thickening $N_t\Gamma$ are defined in [1].

$\mathrm{EL}[C;\Gamma]$ is extremal length with respect to the elastic graph Γ, defined to be

$$\mathrm{EL}[C;\Gamma] \coloneqq \sum_{e \in \mathrm{Edge}(\Gamma)} n_C(e)^2 \cdot \alpha(e),$$

where $n_C(e)$ is the number of times the multi-curve C crosses the edge e in a taut representative of C.

Let C be a simple multi-curve on Σ, i.e., an injective ("simple") map from a union of circles into Σ, considered up to homotopy (or, equivalently, isotopy). Write $\mathrm{EL}[C;\Sigma]$ for extremal length of C with respect to the conformal surface Σ. This is usually written as a a supremum:

$$\mathrm{EL}[C;\Sigma] = \sup_{\rho:\Sigma\to\mathbb{R}_+} \frac{\ell[C;\rho g]^2}{\mathrm{Area}(\Sigma;\rho g)}, \tag{1}$$

where g is a base conformal metric on Σ, and ρ is a rescaling factor. Note that this definition works fine when C is a multi-curve; if $C = \bigcup_i C_i$, then, by definition,

$$\ell[C;g] \coloneqq \sum_i \ell[C_i;g].$$

The supremum in Equation (1) is still obtained by a metric coming from a quadratic differential; in the supremum, each C_i can be thickened into an annulus A_i, and all A_i will have the same width.

Extremal length on surfaces can also be written as an infimum over embeddings of annuli. If the simple multi-curve C has k components, take $A = \bigcup_{i=1}^k A_i$ to be a disjoint union of k annuli. Then

$$\mathrm{EL}[C;\Sigma] = \inf_{\phi:A\hookrightarrow\Sigma} \sum_{i=1}^k \mathrm{EL}(\phi(A_i)) \tag{2}$$

where the infimum runs over all topological embeddings of A into Σ with the core curve of A_i isotopic to C_i. Here, $\mathrm{EL}(\phi(A_i))$ is the extremal length of the image of A_i, more properly defined as the extremal length of A_i with respect to the pull-back by ϕ of the conformal structure on Σ. The extremal length of an annulus A_i is just defined to be $1/\operatorname{mod}(A_i)$, the inverse of the modulus of A_i.

The modulus of an annulus or multi-annulus can in turn be defined by extremal length of a dual family of curves. Given an annulus A_i with a metric g, let $\ell_\|(A_i;g)$ be the shortest

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length of any arc crossing A_i from one boundary component to the other. For a multi-annulus $A = \bigcup_{i=1}^k A_i$, define

$$\ell_\parallel(A; g) := \min_{i=1}^{k} \ell_\parallel(A_i; g),$$

as is natural from the point of view of taking the shortest path of any curve in the family. Then define

$$\mathrm{EL}_\parallel(A) := \sup_{\rho: A \to \mathbb{R}_+} \frac{\ell_\parallel(A; \rho g)^2}{\mathrm{Area}(A; \rho g)}. \tag{3}$$

Lemma 4. *For A any multi-annulus,*

$$\mathrm{EL}(A) = \sum_{i=1}^{k} \mathrm{EL}(A_i) = \frac{1}{\mathrm{EL}_\parallel(A)}.$$

Proof sketch. In Equation (3), the optimal metric ρ for the supremum will have all components A_i having equal transverse length $\ell_\parallel(A_i; \rho g)$, which means that the numerator is the same for all components, and $\mathrm{EL}_\parallel(A)$ is just $1/\sum_i \mathrm{Area}(A_i; \rho g)$, which reduces the result to the case of just one annulus. □

Proposition 5. *Let (Γ, α) be an elastic ribbon graph. Then for C any simple multi-curve on $N\Gamma$, we have*

$$\mathrm{EL}[C; \Gamma] \leqslant t\, \mathrm{EL}[C; N_t\Gamma].$$

Proposition 6. *Let (Γ, α) be an elastic ribbon graph with trivalent vertices, and let $m = \min\{\, \alpha(e) \mid e \in \mathrm{Edge}(\Gamma) \,\}$ be the lowest weight of any edge in Γ. Then, for $t < m/2$ and C any simple multi-curve on $N\Gamma$, we have*

$$t\, \mathrm{EL}[C; N_t\Gamma] < \mathrm{EL}[C; \Gamma] \cdot (1 + 8t/m).$$

Remark 7. The restriction to trivalent graphs in Proposition 6 can presumably be removed with some more work. Since every graph is homotopy-equivalent to a trivalent graph, it is not necessary for our applications. Note that the concrete estimates depend only on the local geometry of Γ, and thus are unchanged under covers.

Proof of Proposition 5. Since $\mathrm{EL}[C; N_t\Gamma]$ is defined as a supremum over all test metrics in the conformal class of the surface $N_t\Gamma$, it suffices to find a test function $\rho\colon N_t\Gamma \to \mathbb{R}_+$ for which

$$EL[C; \Gamma] \leqslant t \frac{\ell[C; \rho g]^2}{\mathrm{Area}(N_t\Gamma; \rho g)},$$

where g is the standard metric on $N_t\Gamma$ (with an edge e corresponding to an $\alpha(e) \times t$ rectangle). Take ρ to be the function which is $n_C(e)$ on the rectangle corresponding to the edge e. Then the shortest representative of $[C]$ will run down the center of each rectangle; since there are n_C different strands running down each rectangle, we have

$$\ell[C; \rho g] = \sum_{e \in \mathrm{Edge}(\Gamma)} (n_C(e))^2 \alpha(e).$$

On the other hand, the area is

$$\mathrm{Area}(N_t\Gamma; \rho g) = \sum_{e \in \mathrm{Edge}(\Gamma)} (\alpha(e) n_C(e)) \cdot (t n_C(e)),$$

so

$$\frac{\ell[C;\rho g]^2}{\operatorname{Area}(N_t\Gamma;\rho g)} = \frac{\left(\sum n_C(e)^2\alpha(e)\right)^2}{t\sum n_C^2\alpha(e)} = \frac{1}{t}\operatorname{EL}[C;\Gamma],$$

as desired. □

Remark 8. Usually, the test function ρ in the proof of Proposition 5 is not optimal (for instance, it is usually not continuous), so we have a strict inequality.

Proof of Proposition 6. For this direction, we need an upper bound on $\operatorname{EL}[C;N_t\Gamma]$, so we will use Equation (2) to calculate extremal length.

We first find suitable annuli. Focus on the thickening N_te of a particular edge e with $n = n_C(e)$ different sections of annuli running along it; we need to divide N_te into pieces corresponding to these different annuli. Divide up the central portion of N_te into $n_C(e)$ horizontal strips of equal height. Inside a $t \times t$ square near each end, we make adjustments so the annuli will glue together well, as sketched in Figure 1. (These squares do not overlap since $t < m/2$.) Specifically, near one end of e, n_1 of the annulus sections will continue to the left-hand neighbor of e at the corresponding vertex, and n_2 will continue to the right-hand neighbor, with $n_1 + n_2 = n$. Divide the interval $[-t/2, 0]$ into n_1 equal sections, divide the interval $[0, t/2]$ into n_2 equal sections, and connect the corresponding annuli.

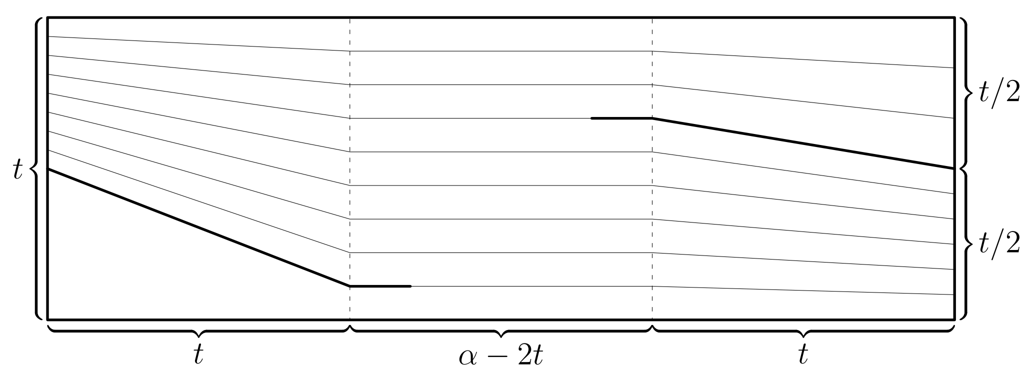

FIGURE 1. The annuli for the estimate in Proposition 6 within the rectangle corresponding to an edge of Γ with elastic weight α. Each portion of an annulus lies within a strip bounded by lines that are horizontal in the middle and get wider or narrower near the ends.

Next we need to given an upper bound on the (total) extremal length of these annuli, which we will do by giving a lower bound on $\operatorname{EL}_{\parallel}(A)$. We do this by again finding a suitable test metric ρg, where g is the standard metric on $N_t\Gamma$ restricted to the annuli.

With this setup, take ρ to be $n_C(e)$ on the central section of each rectangle, and $\sqrt{5}n_C(e)$ on the squares at the ends of the rectangles. In the standard metric g, the vertical width of the annuli is t in the center section and at least $t/2$ in the end squares. In the metric ρg, the vertical width is still t at least $t \cdot \sqrt{5}/2$ in the squares. In the squares, since the edges of the annuli are sloped, the actual width of an arc may be less than the vertical width; but since the slope of the edges is in $[-1/2, 1/2]$, the actual width is at least

$$t \cdot \sqrt{5}/2 \cdot \cos(\tan^{-1}(1/2)) = t.$$

Thus we have

$$\begin{aligned}
\ell_{\|}(A;\rho g) &\geqslant t \\
\operatorname{Area}(A;\rho g) &= \sum_{e\in\operatorname{Edge}(\Gamma)} \bigl(n_C(e)^2(\alpha(e)-2t)t + 2\cdot 5 n_C(e)^2 t^2\bigr) \\
&= \sum_{e\in\operatorname{Edge}(\Gamma)} \bigl(n_C(e)^2\alpha(e)t + 8 n_C(e)^2 t^2\bigr) \\
&= t \sum_{e\in\operatorname{Edge}(\Gamma)} n_C(e)^2\alpha(e)\left(1+\frac{8t}{\alpha(e)}\right) \\
&\leqslant t\,\operatorname{EL}[C;\Gamma]\cdot(1+8t/m) \\
\operatorname{EL}_{\|}(A) &\geqslant \frac{\ell_{\|}(A;\rho g)^2}{\operatorname{Area}(A;\rho g)} \geqslant \frac{t}{\operatorname{EL}[C;\Gamma]\cdot(1+8t/m)} \\
t\,\operatorname{EL}[C;N_t\Gamma] &= t/\operatorname{EL}_{\|}(A) \leqslant \operatorname{EL}[C;\Gamma]\cdot(1+8t/m).
\end{aligned}$$

These inequalities are again far from optimal (the test metric ρg is never continuous), so there is a strict inequality. ☐

References

[1] Dylan P. Thurston, *From rubber bands to rational maps: A research report*, Res. Math. Sci. (2015), arXiv:1502.02561, Accepted.

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