DISCRETE QUADRATIC DIFFERENTIALS

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ABSTRACT. In this short note, we introduce a notion of *discrete quadratic differential* on a filling, elastic graph on a surface. This gives a discrete analogue of holomorphic quadratic differentials, and in particular a system of coordinates on measured foliations. Unlike other representations (e.g., train tracks), this gives a single uniform system of coordinates for all measured foliations on a closed surface. It is also useful for computationally approximating actual harmonic measured foliations.

1. CORNER STRUCTURES AND QUADRATIC DIFFERENTIALS

Definition 1. A *filling elastic graph* (Γ, α) in a surface Σ is a graph Γ (1-dimensional CW complex) embedded in Σ that is *filling*, in the sense that the complementary regions are all disks, and *elastic*, in the sense that each edge e has an associated number $\alpha(e) \in \mathbb{R}_{\geqslant 0}$, the *elastic constant*.

An elastic graph has a natural dual elastic graph (Γ^*, α^*), where Γ^* is the usual dual of Γ and $\alpha^*(e^*) = 1/\alpha(e)$.

Definition 2. A *corner structure* on a filling graph Γ is an assignment of a marking $m(c) \in \{\times, \circ\}$ for each corner c of a face of $\Sigma \backslash \Gamma$. We require that each face have at least two $\times$ markings and each vertex have at least two $\circ$ markings.

A *length structure* (ℓ, m) on a filling elastic graph Γ is a corner structure on Γ and an assignment $\ell(e) \in \mathbb{R}_{\geqslant 0}$ of a *length* to each edge e of Γ.

Example 3. If $f \colon \Gamma \to S^1$ is a map, linear on the edges, so that each vertex has some incident edges going left and some going right, then we can get a length structure by assigning to each edge e with endpoints v_1 and v_2

$$\ell(e) = |f(v_1) - f(v_2)|$$

and, for a corner c at vertex v_2 between vertices v_1 and v_3,

$$m(c) = \begin{cases} \times & \operatorname{sign}(f(v_2) - f(v_1)) \neq \operatorname{sign}(f(v_3) - f(v_2)) \\ \circ & \text{otherwise,} \end{cases}$$

where $f(v_{i+1}) - f(v_i)$ is interpreted in the obvious signed way on the circle. Loosely, $m(c)$ is an $\times$ if there is a sign-change at c when walking around the face.

To construct a length structure from a measured foliation F, overlay Γ on F so that the edges don't backtrack as on the right. Mark a corner with $\circ$ if there is a leaf running in to that corner, and $\times$ otherwise. The length $\ell(e)$ is the total measure of e with respect to the measured foliation.

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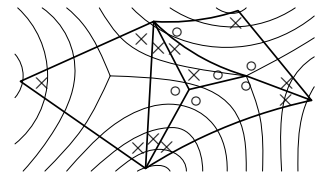
Definition 4. A sequence of non-negative real numbers $(x_1, \ldots, x_k)$ is said to *satisfy the triangle inequality* if, for each $i \in \{1, \ldots, k\}$, we have

$$x_i \leqslant \sum_{j \neq i} x_j.$$

Observe that satisfying the triangle inequality implies that $k \geqslant 2$. If $k = 2$, then $x_1 = x_2$. If $k = 3$, this is the usual triangle inequality.

Definition 5. A length structure (ℓ, m) on an elastic graph (Γ, α) is *closed* if it satisfies the following triangle inequality on each face f of Γ. Around f, we see a (cyclic) sequence of edges $e_1, \ldots, e_n$, separated by corners $c_0, \ldots, c_n = c_0$ (with c_i between e_{i-1} and e_i). Rotate the face so that $m(c_0) = \times$, and suppose that there are a total of k different $\times$'s around the face, at locations

$$0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = n.$$

Then, for each sequence of edges between $\times$'s, we compute the total length:

$$x_j = \sum_{t=i_{j-1}+1}^{i_j} \ell(e_t).$$

The constraint to make the length structure closed is that $(x_1, \ldots, x_k)$ satisfy the triangle inequality of Definition 4.

Note that, if we hadn't required that there be at least two $\times$'s around a face, it would be forced by the triangle inequalities in Definition 4. In the case where there are exactly two $\times$'s, the two inequalities of Definition 4 combine to give an equality $x_1 = x_2$.

Example 6. The length structures from Example 3 are automatically closed: On each face, we can divide the edges into "left-moving" edges and "right-moving" edges, with the sum of lengths of the left-moving edges equal to the sum of lengths of the right-moving edges. Left-moving and right-moving edges are divided by an $\times$. This immediately implies the relevant triangle inequalities.

Definition 7. A length structure (ℓ, m) on an elastic graph (Γ, α) induces a dual length structure ℓ^* on the dual graph by

$$\ell^*(e^*) = \ell(e)/\alpha(e)$$

$$m^*(c^*) = \begin{cases} \times & m(c) = \circ \\ \circ & m(c) = \times, \end{cases}$$

where e and e^* are corresponding edges and c and c^* are corresponding corners. Observe that the double dual returns to the original length structure.

Definition 8. A length structure is *co-closed* if its dual is closed; that is, it satisfies weighted triangle inequalities at each vertex.

Example 9. Suppose Γ is a train track on the surface. The train track structure gives a corner structure on Γ: At each switch of Γ, place $\circ$'s at the two smooth corners, and $\times$'s everywhere else. If Γ is a weighted train track with weights $\mu(e)$, we get a co-closed length structure on Γ, by setting $\ell(e) = \alpha(e)\mu(e)$. The triangle inequalities at the vertices become the usual equality constraints for weights on train tracks.

Definition 10. A *harmonic* length structure or *discrete quadratic differential* on a filling elastic graph is a closed and co-closed length structure on the graph.

One intuition is that a discrete quadratic differential (non-zero on each edge) gives a rectangle-tiled surface: each edge e gives a rectangle of aspect ratio $\alpha(e)$, with length $\ell(e)$ and width $\ell^*(e^*) = \ell(e)/\alpha(e)$. These rectangles are sewn together at the faces and vertices. The triangle inequalities guarantee that there is at least one way to sew them together, although there may be some ambiguity; this will be addressed later.

As with usual holomorphic quadratic differentials, there is a version of the Poincaré-Hopf theorem.

Definition 11. Given a corner structure m, let $n_\times$ and $n_\circ$ be the number of $\times$'s or $\circ$'s in m around a face or vertex. For a face f and vertex v, define
$$\text{index}(f) := n_\times(f) - 2$$
$$\text{index}(v) := n_\circ(V) - 2.$$

Proposition 12. *For any length structure on a graph Γ in a surface Σ,*
$$\sum_{f \ a \ face} \text{index}(f) + \sum_{v \ a \ vertex} \text{index}(v) = -2\chi(\Sigma).$$

Proof. Let V, E, and F be, respectively, the number of vertices, edges, and faces of Γ. Since the total number of corners is $2E$, we have
$$2E = \sum_f n_\times(f) + \sum_v n_\circ(v)$$
$$= 2F + \sum_f \text{index}(f) + 2V + \sum_v \text{index}(v). \qquad \square$$

In light of Proposition 12, we say that a vertex or face is *non-singular* if its index is 0, i.e., if it has the minimal number of $\times$'s or $\circ$'s, respectively.

We would like to say that a discrete quadratic differential has associated horizontal and vertical measured foliations. However, we need a little bit of extra structure.

Definition 13. A *discrete measured foliation* on a filling graph $\Gamma \subset \Sigma$ is a closed length structure on Γ, together with, for each face f of Γ, a choice of tree T_f and map $\phi_f \colon \partial f \to T_f$ so that

- between adjacent $\times$'s on ∂f, the map ϕ_f is an isometry (with respect to the metric on Γ given by the length structure), and
- T_f is the convex hull of $\phi_f(v)$, where v ranges over the vertices of f that are marked by $\times$.

If f is non-singular, then T_f is necessarily an interval. If f has index 1, then T_f is a tripod, with lengths uniquely determined by the length structure. In general, the triangle inequalities guarantee there is at least one valid tree T_f. For $\text{index}(f) \geqslant 1$, the set of possible trees generically has dimension $\text{index}(f) - 1$, and may be conveniently parameterized by triangulating the face and specifying the lengths on each edge of the triangulation, subject to triangle inequalities.

If the length of every edge is non-zero, a discrete measured foliation canonically determines a measured foliation on Σ. (If some of the edge lengths are 0, then you canonically get a partial measured foliation, which may be blown up to get an actual measured foliation.)

Proposition 14. *For any filling graph* Γ, *every Whitehead equivalence class of measured foliations may be realized by at least one discrete measured foliation on* Γ.

We can also see Whitehead equivalence discretely. For simplicity, we assume that Γ is a triangulation, i.e., each face of Γ has three sides; this guarantees that we do not need the extra structure of the trees T_f.

Proposition 15. *Two discrete measured foliations on a triangulation* Γ *give Whitehead equivalent measured foliations on* Σ *if and only if they are related by the following local moves, in each case constrained to preserve the triangle inequalities.*

(1) *At a non-singular vertex, add a constant to the lengths on one side of the vertex and subtract the same constant from the lengths on the other side.*
(2) *Change an* ○ *to an* ×, *or vice versa.*
(3) *Around an edge of length 0 with adjacent markings* ×○×○ *in counterclockwise order, switch the markings to* ○×○×, *or vice versa.*

Definition 16. The *energy* of a length structure (ℓ, m) on an elastic graph (Γ, α) is

$$E(\ell) := \sum_e \frac{\ell(e)^2}{\alpha(e)}.$$

Proposition 17. *If* $\Gamma \subset \Sigma$ *is a filling elastic graph, then every discrete measured foliation is equivalent by the moves above to an essentially unique harmonic (co-closed) measured foliation. Furthermore, this harmonic measured foliation is a global minimum for the energy.*

Here *essentially unique* means that the lengths are the same, and that the markings differ by only the two allowed moves that don't change the lengths at all. One proof goes through an explicit algorithm: With fixed markings, energy is a quadratic function of the edge lengths. The triangle inequalities (and positivity of the lengths) give linear conditions on the lengths. The minimum with given markings is therefore a quadratic-programming problem, which is efficiently solvable. You may then need to switch around some markings in order to make further progress, but any local minimum is necessarily harmonic.

From the point of view of surfaces, one interesting feature is that you get a *canonical* representative of *any* measured foliation, including for closed surfaces. Train tracks, by contrast, give canonical representatives for measured foliation (assuming that the complementary regions all have at least 3 cusps, i.e., that every face is singular), but a single train track does not cover all measured foliations.

2. Train-track paths and recurrence

Definition 18. A *train track path* in a corner structure m for a filling graph Γ is a path γ of edges of Γ so that, at each vertex that γ passes through, there is at least one ○ on each side of γ through the vertex.

(Picture of train-track condition.)

Lemma 19. *If* (ℓ, m) *is a discrete measured foliation on* Γ *and* γ *is a closed train track path on* (Γ, m), *then the length of* γ *with respect to the measured foliation induced by* (ℓ, m) *is*

$$\sum_{e \in \gamma} \ell(e).$$

Definition 20. A corner structure m on Γ is *recurrent* if, for every edge of Γ, there is a closed train track path that passes through e. Dually, m is *transversally recurrent* if (m^*, Γ^*) is recurrent.

Theorem 21. *The corner structure (Γ, m) is transversally recurrent if and only if there is a closed length structure compatible with m that is positive on each edge.*

This is very close to Proposition 1.3.1 and Corollary 1.3.5 by Penner and Harer [5], although the train track structures he works with are less general (in that all vertices in a standard train track structure are non-singular in our terms). The statement can also be strengthened slightly to include cases where some edges have zero length.

Proof. If m is transversally recurrent, let (γ_i) be a sequence of loops that pass through each edge. Then define
$$\ell(e) = \sum_i n(\gamma_i, e),$$
where $n(\gamma_i, e)$ is the number of times the loop γ_i crosses the edge e. These satisfy the triangle inequalities on each face.

Conversely, suppose that we have a closed length structure ℓ on (Γ, m). The conditions for existence of a closed length structure are linear equalities and inequalities with rational coefficients, so we may assume that ℓ is rational. Scale ℓ so that the values are even integers. Then the triangle inequalities guarantee that there is an integer measured foliation F (i.e., a collection of curves) so that $\ell(e)$ is the number of intersections of F with e. But then for each edge e there is at least one curve that intersects e. $\qquad\square$

Corollary 22. *If (ℓ, m) is a harmonic discrete quadratic differential on Γ, then, for each edge e with $\ell(e) \neq 0$, there is a closed train track path running through e.*

Proof. Direct from Theorem 21, using the fact that ℓ is co-closed. $\qquad\square$

3. COMPUTATIONAL APPROXIMATIONS

There are two obvious types of applications for this model for discrete quadratic differentials:

- You could take as coarse a graph as possible, with very few edges, to give (e.g.) an efficient coding for the space of measured foliations.
- You could take a very fine graph, with the goal of approximating well the actual harmonic measured foliations on the underlying surface.

We now explore the second of these. There is a long history of discrete approximations to harmonic functions or 1-forms on planar domains or Riemann surface Σ.

The most relevant family of approximations starts from approximating Σ by a triangulation T into Euclidean triangles with acute angles. Then an arbitrary function on Σ can be approximated by a PL function f, linear on each simplex. Harmonic functions are the minimum of the Laplacian ∇^2, which can be evaluated on f, despite the fact that f is not differentiable. After a short calculation [1,3,6], one finds that

$$(23) \qquad \nabla^2(f) = \sum_{e \text{ edge of } T} \frac{\cot(\alpha) + \cot(\beta)}{2} \big(f(v) - f(w)\big),$$

where v and w are the two endpoints of e and α and β are the two angles opposite from e in the adjacent triangles. Since the triangles were assumed to be acute, these coefficients are

FIGURE 1. Harmonic measured foliations computed by this method. The measured foliation obtained by taking a measured foliation supported in a neighborhood of three disjoint curves (on the left, middle, and right of the surface) and finding the harmonic representative in the Whitehead equivalence class.

positive. (In fact, we just need $\alpha + \beta < \pi$ for the coefficients to be positive.) By comparison with Definition 16, it is natural to look at the filling elastic graph made of the 1-skeleton of T, with

$$(24) \qquad \alpha(e) = \frac{2}{\cot(\alpha) + \cot(\beta)}$$

with notation as above.

There are very precise theorems about how well discrete harmonic 1-forms, i.e., 1-forms that are local minima for Equation (23), approximate actual harmonic 1-forms.

David Palmer, in work supervised by Steven Gortler, refined and implemented the above algorithm for triangulations with cotangent weights, with the following broad steps.

(1) Start with a surface Σ embedded in $\mathbb{R}^3$.
(2) Take a set of points well-distributed on Σ and connect nearby points to get a triangulation T of Σ. If necessary, adjust the triangulation so the coefficients in the next step are positive.
(3) Create an elastic graph (Γ, α) from the 1-skeleton of T, with α from Equation (24).
(4) To create an initial (non-harmonic) foliation, pick a curve C on Σ transverse to the edges of T and not entering and exiting the same side of a triangle of T.
(5) Create a foliation F_0 so that the length of an edge e is equal to the size of $C \cap e$. (Typically this is 0 or 1.)
(6) Use an iterated relaxation technique to find the harmonic foliation F_1 in the Whitehead class of F_0.

Details will appear in David Palmer's forthcoming thesis. Figure 1 shows an example of the output from his program.

The cotangent weights can be generalized considerably into decompositions into quadrilaterals where the diagonals meet at right angles, considered by Mercat [4]. To construct such a decomposition into quadrilaterals from a triangulation T by acute triangles, take the dual decompositions T^*, with the vertices of T^* placed at the circumcenters of the triangles of T,

and join each vertex of T^* to the vertices of the corresponding triangle of T. The important special case where the quadrilaterals are rhombi was already considered by Duffin [2].

It is worth noting that the notion of a discrete quadratic differential is self-dual: if (ℓ, m) is a discrete quadratic differential on (Γ, α), then (ℓ^*, m^*) is a discrete quadratic differential on (Γ^*, α^*). This allows one to get computational approximations to the harmonic conjugate of a given measured foliation.

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