HYPERBOLIC VOLUME AND THE JONES POLYNOMIAL: A CONJECTURE

Conjecture 1 (Kashaev-Murakami-Murakami). *For any hyperbolic knot K,*

$$2\pi \times \lim_{n\to\infty} \frac{\log|J'_n(K; e^{2\pi i/n})|}{n} = \mathrm{vol}(S^3 \setminus K)$$

where

- $J_n(K; q)$ is the coloured Jones polynomial of K, the quantum invariant associated to $\mathrm{S}U(2)$ in the representation of dimension n. (It is a linear combination of the original Jones polynomial on cablings of the knot with up to $n-1$ parallel copies, in the same way that the n dimensional representation X_n of $\mathrm{S}U(2)$ is a linear combination of $X_2^{\otimes k}$ for k at most $n-1$.) $J_n(K; q)$ is a power of $q^{1/2}$ times a polynomial in q.
- $$J'_n(K; q) = \frac{J_n(K; q)}{[n]} \qquad [n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$
  J'_n is a Laurent polynomial in q.
- $\mathrm{vol}(S^3 \setminus K)$ is the volume of the unique complete hyperbolic Riemannian metric on the knot complement.

This conjecture provides a connection between the quantum invariants of K and its classical geometry, which has been missing until now. A consequence is that any knot with trivial Vassiliev invariants is the unknot.

The above is the minimal form of the conjecture. There are several extensions, which I will only mention:

- to non-hyperbolic knots. The appropriate analogue of the RHS is the simplicial or Gromov norm of the complement, which turns out to be the sum of the volumes of the hyperbolic pieces in the natural JSJ decomposition of the complement.
- to links.
- to manifolds with torus boundary.

The conjecture is known to be true for torus knots (both sides are 0) and for the figure eight knot. This lecture will review the proof for the figure eight knot and indicate how one might approach the general case.

1. Hyperbolic geometry refresher

How does one compute the hyperbolic volume of a knot complement? One way to compute volumes in general is to cut your manifold up into pieces; say, take a triangulation and compute the volume of each tetrahedron. Finite tetrahedra have many degrees of freedom (they may be parametrized by the edge lengths). In hyperbolic space, there is a better way: use *ideal tetrahedra*. If you send the 4 vertices of a tetrahedron off to infinity in hyperbolic space, the volume remains bounded! We will decompose our knot complement into ideal tetrahedra.

The ideal tetrahedron has 4 vertices off at infinity in various directions, on S^2_∞, which is the same as $\mathbb{C}P^1$ with the natural action of $PSL_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$. Any 3 points on $\mathbb{C}P^1$ are conjugate, so we may take 3 of the vertices to be at 0, 1, and $\infty \in \mathbb{C}P^1$. The remaining point will be at some $\lambda \in \mathbb{C}$; this λ will parametrize the tetrahedron Δ_λ.

Proposition 1.

$$\mathrm{vol}(\Delta_\lambda) = \mathrm{im}(\mathrm{Li}_2(\lambda)) + \arg(1-\lambda)\log|\lambda|$$

where

$$\mathrm{Li}_2(\lambda) = -\int_0^\lambda \frac{\log(1-\lambda)}{\lambda}\,d\lambda = \sum_{n\geq 0} \frac{\lambda^n}{n!}$$

is called the dilogarithm *function.*

Now we need to understand how to glue ideal tetrahedra to get $S^3 \setminus K$. The easiest way to think about this is to trim the corners, so that you are gluing truncated tetrahedra; the gluing will then yield a manifold with torus boundary. For instance, two tetrahedra may be glued as indicated on the attached sheet to get the figure eight knot complement.

To get a hyperbolic structure on the complement, there are some algebraic equations (the matching equations) on the parameters λ_i of the tetrahedra:

- 1 equation per edge, so that, e.g., the dihedral angles will sum up to 2π; and
- 1 equation at the boundary so that the boundary torus will be flat or, equivalently, so that the hyperbolic structure will be complete.

For the decomposition of $S^3 \setminus$ (fig 8 knot) above, these equations imply that the two tetrahedra are both regular ideal tetrahedra, with $\lambda = e^{\pm\pi i/3}$. (To see that these tetrahedra are regular, note that 0, 1, and $e^{\pi i/3}$ lie on the vertices of a regular triangle in $\mathbb{C}$, so these three vertices are symmetric; combined with symmetries that all ideal tetrahedra have, this implies that all the vertices are symmetric.)

2. Computations for (fig 8 knot)

Consider the formula for V_n((fig 8 knot)) shown on the slide. It is of the general form

$$V_n = \sum_k f_n(k) \qquad \text{where} \qquad f_n(k) > 0.$$

We see that

$$\max_k f_n(k) \leq V_n \leq n \max_k f_n(k)$$

so that the asymptotic exponential growth of V_n is the same as the growth of $\max_k f_n(k)$.

To find the maximum of $f_n(k)$, we set the ratio of adjacent terms to 1:

$$1 = \frac{f_n(k)}{f_n(k-1)} = (1-q^k)(1-q^{-k}) \quad \Rightarrow \quad q^k = e^{\pm\pi i/3} \quad \Leftrightarrow \quad k \approx \frac{n}{6}, \frac{5n}{6}$$

To compute the asymptotic growth of this maximum, we need to compute the asymptotics of $\lceil k \rceil!$. We can turn the product into a sum and then approximate it by an integral:

$$\begin{aligned}\lceil k \rceil! &= \prod_{l=1}^{k}(1-q^l) \\ &= \exp\left(\sum_{l=1}^{k}\log(1-q^l)\right) \\ &\approx \exp\left(\frac{N}{2\pi i}\int_1^{q^k}\frac{\log(1-x)}{x}\,dx\right) \\ &= \exp\left(\frac{N}{2\pi i}(\mathrm{Li}_2(q^k)-\frac{\pi^2}{6})\right).\end{aligned}$$

We then have

$$f_n(k) = |\lceil k \rceil!|^2 \approx \exp\left(-\frac{N}{2\pi}\times 2\,\mathrm{im}(\mathrm{Li}_2(q^k))\right)$$

and

$$v_n \approx f_n(\frac{5n}{6}) \approx \exp\left(\frac{N}{2\pi}\times 2\,\mathrm{im}(\mathrm{Li}_2(e^{\pi i/3}))\right) = \exp\left(\frac{N}{2\pi}\times \mathrm{vol}(S^3\setminus(\text{fig 8 knot}))\right)$$

as desired.

3. Further comments

- In general, the formulas for the coloured Jones polynomial involve sums of products of quantum factorials. Each quantum factorial contributes an Li_2 (more or less the volume of an ideal tetrahedron) to the asymptotics. To maximize the product, we set the ratio of adjacent terms to 1; this boils down to solving some algebraic equation for the q^k. (k is a variable to be summed over.) This is the correct general form to solve the matching equations and compute the hyperbolic volume of the complement.[1]
- In small examples, you can make this work precisely, with one quantum factorial per tetrahedron in the simplest decomposition and the same algebraic matching equations. We saw above how to do it for the figure eight knot; it can also be done for the knots 5_2 and 6_1, which decompose into 3 and 4 tetrahedra respectively.
- More generally, the R-matrix formulation gives a formula for the coloured Jones polynomial with 5 quantum factorials per planar crossing. (See the slide.) There is a corresponding decomposition of the complement of the knot into ideal tetrahedra with 5 tetrahedra per crossing. (Many of these ideal tetrahedra are degenerate.) The gluing equations are the same on the two sides.
- So why is this still a conjecture? The problem is that the terms in the sum are not generally > 0 and that the solutions to the gluing equation are not generally on the unit circle $|\lambda_i| = 1$. Since the summation does not pass through the critical point, we must make some additional argument to show that we can deform the contour

[1] There are also terms like q^{kl} which contribute factors like $\log q^k \log q^l$ to the asymptotics; these become the extra term in the hyperbolic volume $\log|\lambda| \arg(1-\lambda)$ at the critical point.

appropriately. The major obstacle to proving the conjecture from this point of view is to show that the deformed contour actually passes through the desired critical point.

- To see that this can be a real problem, note that we can make similar arguments for the coloured Jones polynomial $J_n(e^{2\pi i/r}$ with n/r fixed or for the Witten-Reshetikhin-Turaev invariants of a closed 3-manifold M. In both cases, the same optimistic formal asymptotics predict that the invariants will grow exponentially, in the WRT case as the volume of the hyperbolic structure on M. But these invariants actually grow at most polynomially.