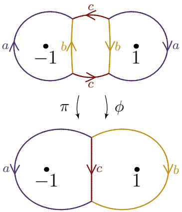
Detecting rational maps using elastic graphs

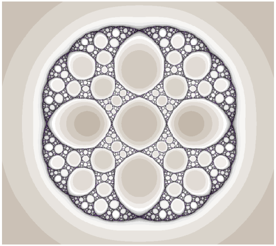
Dylan Thurston

In Memoriam: William P. Thurston, 1946–2012

Portions joint with J. Kahn and K. Pilgrim

`http://pages.iu.edu/~dpthurst/writing/DetectRational.pdf`





August 18, 2015

Outline

Spines for branched self covers

Branched self-cover of sphere

$$f\colon (S^2, P) \to (S^2, P).$$

Gives *virtual endomorphism* or *topological automaton* of a spine:

$$\begin{tikzcd}
S^2 \setminus f^{-1}(P) \arrow[r, shift left, "\pi_S"] \arrow[r, shift right, "\phi_S"'] & S^2 \setminus P \\
\Gamma_1 \arrow[u, hook] \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\phi"'] & \Gamma_0 \arrow[u, hook]
\end{tikzcd}$$

with π_S a covering map (restriction of f), ϕ_S the inclusion map, and maps π and ϕ on graphs commuting up to homotopy.

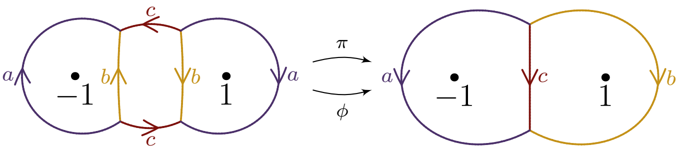
Spine for Σ: graph that fills Σ (complement: punctured disks)

Example: $f(z) = (1 + z^2)/(1 - z^2)$

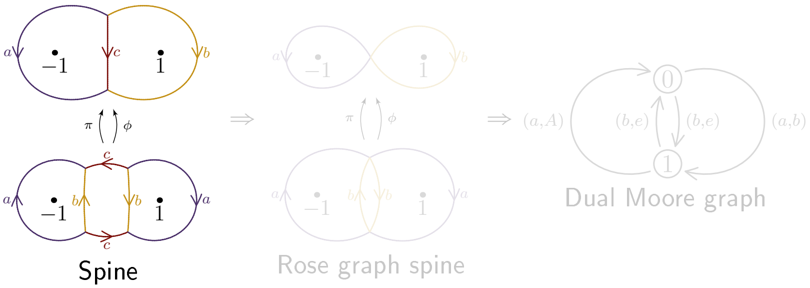
Critical portrait:

$$0 \xrightarrow{(2)} 1 \longrightarrow \infty \underset{}{\overset{(2)}{\rightleftarrows}} -1$$

Set $P = \{-1, 1, \infty\}$

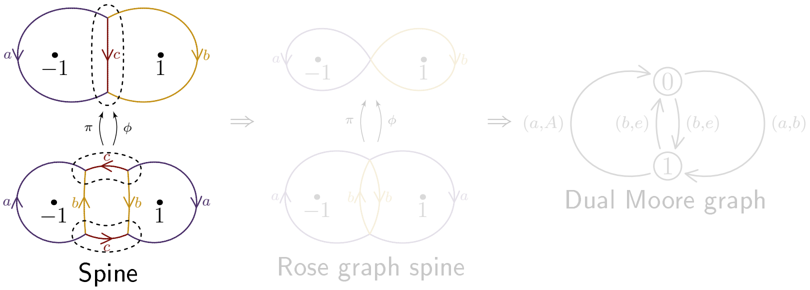


Spines to automata



- Collapse maximal tree in Γ_0 and lift in Γ_1 to get R_0, R_1
- Homotop ϕ so vertices of R_1 map to vertex of R_0
- Label edges by $\pi_1(R_0)$

Spines to automata



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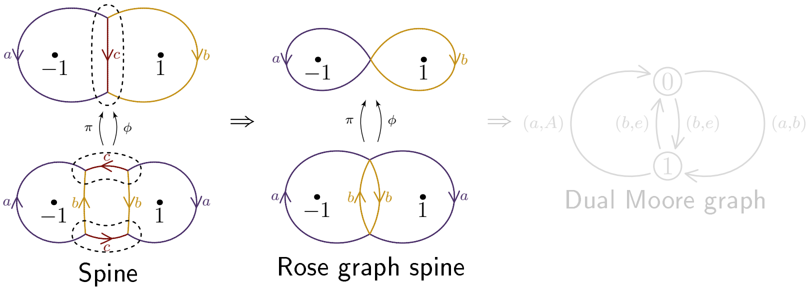

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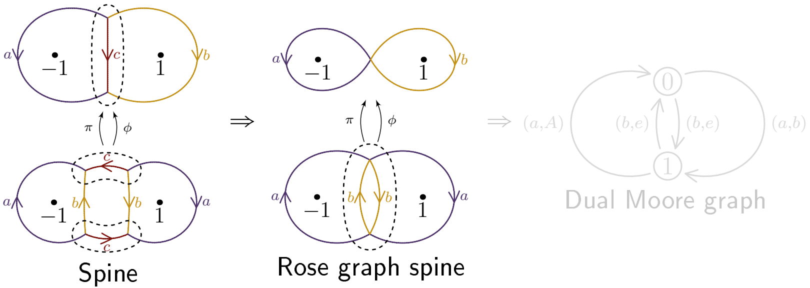

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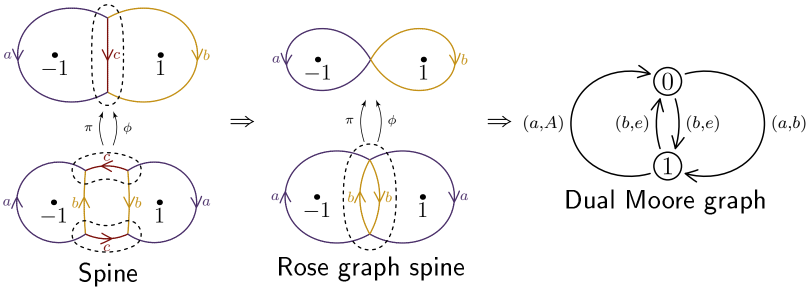

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Iteration

$$
\begin{tikzcd}
 & & \Gamma_2 \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dll, bend right, "\phi_2"'] \arrow[drr, bend left, "\pi_2"] & & \\
 & \Gamma_1 \arrow[dl, "\phi"'] \arrow[dr, "\pi"] & & \Gamma_1 \arrow[dl, "\phi"'] \arrow[dr, "\pi"] & \\
\Gamma_0 & & \Gamma_0 & & \Gamma_0
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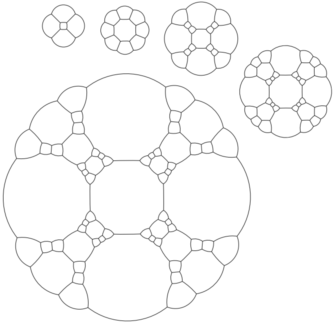
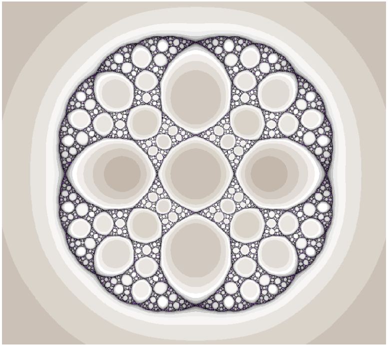




Outline

Spines and automata

▸ Main theorem

Energies

Behavior under iteration

Questions

Main theorem

Theorem

Let $f\colon (S^2, P) \circlearrowleft$ be a branched self-cover with at least one branch point in each cycle in P, and let $\pi, \phi\colon \Gamma_1 \to \Gamma_0$ be a corresponding virtual endomorphism. Then the following are equivalent:

- *f equivalent to a rational map*
- *for some metric on Γ_0 and some $n > 0$, $\phi_n\colon \Gamma_n \to \Gamma_0$ is loosening*
- *for every metric on Γ_0 and every $n \gg 0$, $\phi_n\colon \Gamma_n \to \Gamma_0$ is loosening*

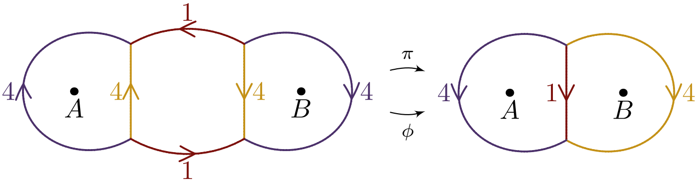
Definition

If Γ_1, Γ_0 are metric graphs, a Lipshitz map $\phi\colon \Gamma_1 \to \Gamma_0$ is *loosening* if, for almost every $y \in \Gamma_0$,

$$\sum_{x \in \phi^{-1}(y)} |\phi'(x)| < 1.$$

In particular, ϕ is 1-Lipshitz.

Example: $f(z) = (1 + z^2)/(1 - z^2)$



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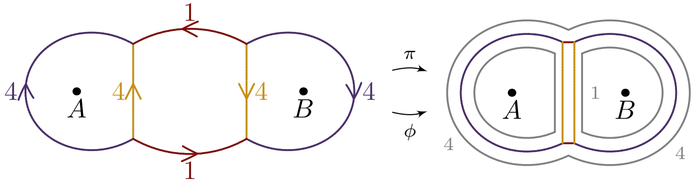

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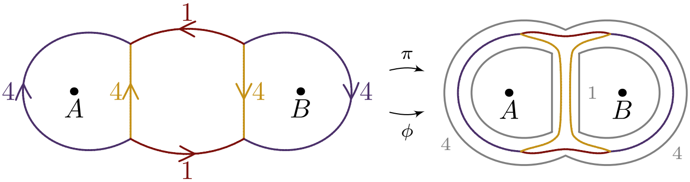

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\end{tikzcd}
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- Elastic graph: graph with *spring constant* on each edge
- Length graph: graph with *length* on each edge
- $\mathrm{Emb}(\phi) = \sup_{y \in \Gamma_2} \sum_{x \in \phi^{-1}(y)} |\phi'(x)|$
- $\mathrm{Dir}(f) = \int_{x \in \Gamma_2} |f'(x)|^2 \, dx$
- $\mathrm{EL}(c) = \int_{y \in \Gamma_1} n_c(y)^2 \, dy$

Submultiplicative. . .

- $\mathrm{Dir}(f \circ \phi) \leqslant \mathrm{Emb}(\phi)\,\mathrm{Dir}(f)$
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. . . minimizing over homotopy class:

- $\mathrm{Dir}[f \circ \phi] \leqslant \mathrm{Emb}[\phi]\,\mathrm{Dir}[f]$
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Stretch factors for graph maps

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- For $f\colon \Sigma_2 \to K$, have *Dirichlet energy*.
- For $c\colon C \to \Sigma_1$ an embedded simple closed multi-curve, have *extremal length* $\mathrm{EL}[c]$.
- For $\phi\colon \Sigma_1 \hookrightarrow \Sigma_2$ a topological embedding, $\mathrm{SF}[\phi] = \sup_{C,c} \dfrac{\mathrm{EL}[\phi \circ c]}{\mathrm{EL}[c]}$.

Theorem (Kahn, Pilgrim, T)

$\mathrm{SF}[\phi] \leqslant 1 \Leftrightarrow \phi$ *homotopic to conformal embedding*
$\mathrm{SF}[\phi] < 1 \Leftrightarrow \phi$ *homotopic to conformal embedding with some space*

If $\mathrm{SF}[\phi] > 1$, then $\mathrm{SF}[\phi]$ is the minimal quasi-conformal constant in $[\phi]$.
General interpretation??

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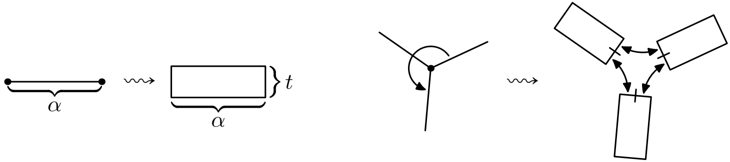
Relating graphs and surfaces

Elastic ribbon graph $\Gamma \rightsquigarrow$ Conformal surface $N_t\Gamma$



Proposition

For Γ an elastic ribbon graph, $t \ll 1$, and c a curve on Γ

$$\mathrm{EL}_\Gamma[c] \leqslant t \cdot \mathrm{EL}_{N_t\Gamma}[c] \leqslant (1+\varepsilon)\,\mathrm{EL}_\Gamma[c],$$

where ε depends only on the local geometry of Γ.

Corollary

For $\phi\colon \Gamma_1 \to \Gamma_2$ a suitable map between elastic ribbon graphs and $t \ll 1$,

$$(1-\varepsilon)\,\mathrm{SF}[\phi] \leqslant \mathrm{SF}[N_t\phi] \leqslant (1+\varepsilon)\,\mathrm{SF}[\phi]$$

where ε depends only on the local geometry of Γ_1 and Γ_2.

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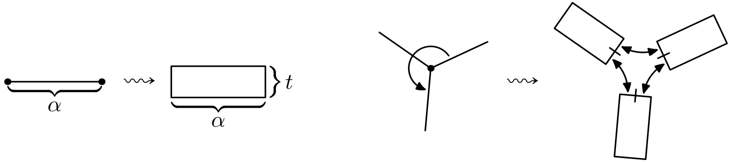

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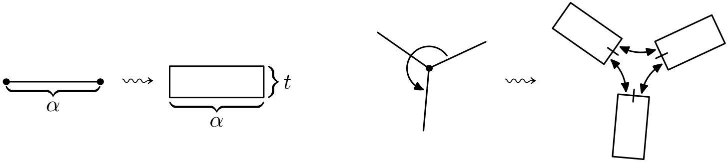
where ε depends only on the local geometry of Γ_1 and Γ_2.

Relating graphs and surfaces

Elastic ribbon graph Γ $\rightsquigarrow$ Conformal surface $N_t\Gamma$



Proposition

For Γ an elastic ribbon graph, $t \ll 1$, and c a curve on Γ

$$\mathrm{EL}_\Gamma[c] \leqslant t \cdot \mathrm{EL}_{N_t\Gamma}[c] \leqslant (1+\varepsilon)\,\mathrm{EL}_\Gamma[c],$$

where ε depends only on the local geometry of Γ.

Corollary

For $\phi\colon \Gamma_1 \to \Gamma_2$ a suitable map between elastic ribbon graphs and $t \ll 1$,

$$(1-\varepsilon)\,\mathrm{SF}[\phi] \leqslant \mathrm{SF}[N_t\phi] \leqslant (1+\varepsilon)\,\mathrm{SF}[\phi]$$

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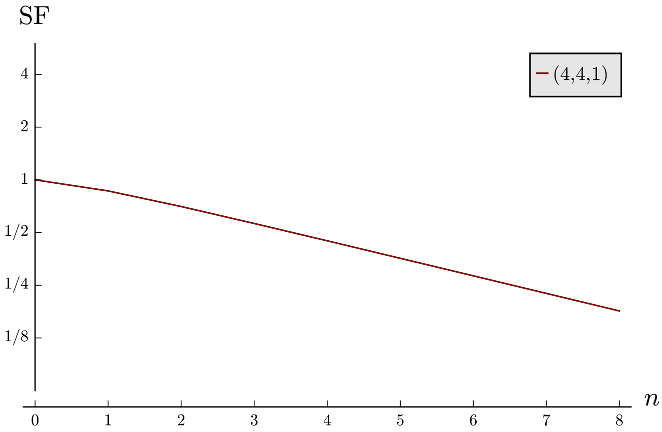
Outline

Experimental data



Logarithmic plot of SF, iterating running example with varying elastic lengths and topology.

Dashed line shows limiting behaviour: $\mathrm{SF}[\phi_n] \geqslant 2^{-n/3}$ in this case.

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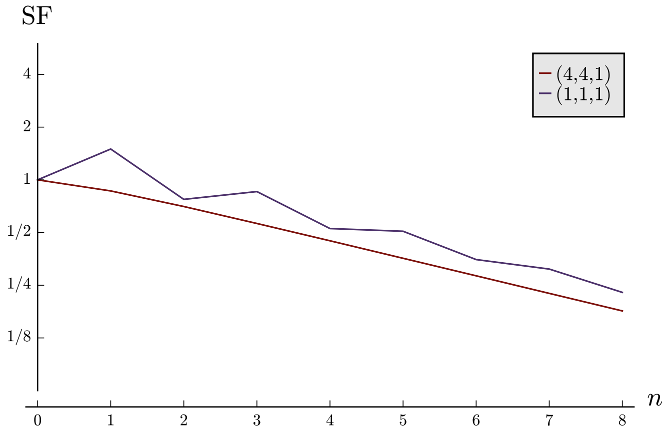

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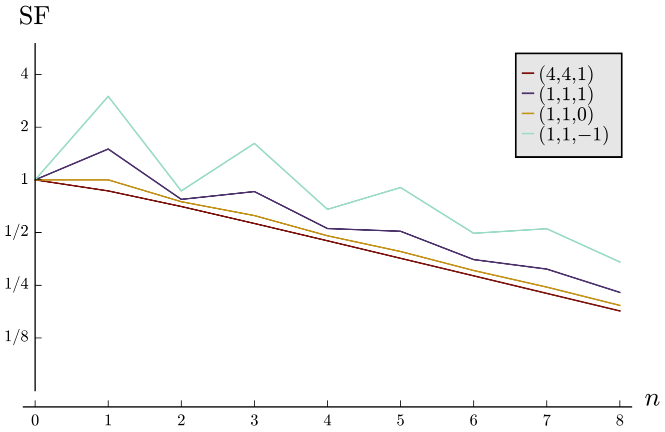

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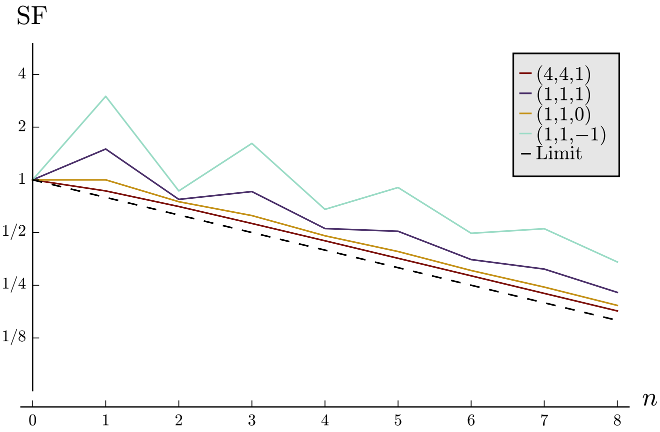

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Key to main theorem: Asymptotics of stretch factors

Definition

For $\pi, \phi\colon \Gamma_1 \to \Gamma_0$ a virtual endomorphism of elastic graphs, the *asymptotic stretch factor* is

$$\overline{\mathrm{SF}}[\phi] = \lim_{n\to\infty} \sqrt[n]{\mathrm{SF}[\phi_n]}.$$

Lemma

$\overline{\mathrm{SF}}[\phi]$ is independent of elastic weights on Γ_0.

Proof.

$$\mathrm{SF}[\phi \circ \psi] \leqslant \mathrm{SF}[\phi]\,\mathrm{SF}[\psi]$$

$$\begin{aligned}\mathrm{SF}[\Gamma'_n \to \Gamma'_0] &\leqslant \mathrm{SF}[\Gamma'_n \to \Gamma_n]\,\mathrm{SF}[\Gamma_n \to \Gamma_0]\,\mathrm{SF}[\Gamma_0 \to \Gamma'_0] \\ &= K_1\,\mathrm{SF}[\Gamma_n \to \Gamma_0] K_2.\end{aligned}$$

□

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Asymptotic stretch factor for surfaces

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For $\pi, \phi\colon S_1 \to S_0$ a virtual endomorphism of surfaces, the *asymptotic stretch factor* is

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Lemma

$\overline{\mathsf{SF}}_{\mathrm{Surf}}[\phi]$ *is independent of conformal structure on S_0.*

Lemma

$\overline{\mathsf{SF}}_{\mathrm{Surf}}[\phi] = \overline{\mathsf{SF}}_{\mathrm{Graph}}[\phi]$.

Proof.

Use same technique as above, combined with result on approximating SF for Γ and $N_t\Gamma$. $\square$

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Completing the proof

Theorem

Let $f\colon (S^2, P) \circlearrowleft$ be branched self-cover, at least one branch point in each cycle in P. Then f equivalent to a rational map $\Leftrightarrow$ exists surface $\Sigma_0 \subset S^2 \setminus P$ so that

$$\begin{tikzcd} \Sigma_1 \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\phi"'] & \Sigma_0 \end{tikzcd}$$

with ϕ a conformal embedding.

Proof.

Folklore, using quasi-conformal surgery; written by Cui–Peng–Tan. □

Corollary

$\overline{\mathrm{SF}}_{\mathrm{Surf}}[\phi] < 1$ *iff* (π, ϕ) *is equivalent to a rational map*

Proof.

Combine theorem above with characterization of surface embedding by SF. □

Completing the proof, cont

Corollary

$\overline{\mathsf{SF}}_{\mathrm{Surf}}[\phi] < 1$ *iff* (π, ϕ) *is equivalent to a rational map*

Theorem

Let $f\colon (S^2, P) \circlearrowleft$ *be a branched self-cover with at least one branch point in each cycle in* P*, and let* $\pi, \phi\colon \Gamma_1 \to \Gamma_0$ *be a corresponding virtual endomorphism. Then the following are equivalent:*

- f *equivalent to a rational map*
- *for some metric on* Γ_0 *and some* $n > 0$, $\phi_n\colon \Gamma_n \to \Gamma_0$ *is loosening*
- *for every metric on* Γ_0 *and every* $n \gg 0$, $\phi_n\colon \Gamma_n \to \Gamma_0$ *is loosening*

Outline

Questions

- Census of rational maps? (Table of 464 rational maps vs. 1,701,935 prime knots)
- Apply criterion in concrete cases (e.g., matings)?
- Polynomial-sized certificates?
- Is $\overline{\mathrm{SF}}[\phi]$ always algebraic? How to compute it?
- New proof of W. Thurston's annular obstruction?
- Direct interpretation of $\mathrm{SF}[\phi]$ for $\phi\colon S_1 \to S_2$ when $\mathrm{SF}[\phi] < 1$? (When $\mathrm{SF}[\phi] \geqslant 1$, it equals the minimal quasi-conformal dilatation.)
- What happens for topological automata that do not come from rational maps? What does $\overline{\mathrm{SF}}[\phi] < 1$ mean in general?