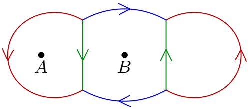
Rubber Bands, Square Tilings, and Rational Maps

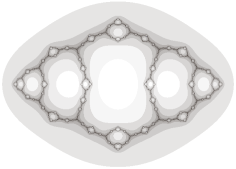
Dylan Thurston

In Memoriam: William P. Thurston, 1946–2012

Portions joint with J. Kahn and K. Pilgrim

http://pages.iu.edu/~dpthurst/writing/DetectRational.pdf



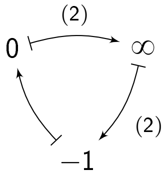


September 20, 2014

Rational maps: An example

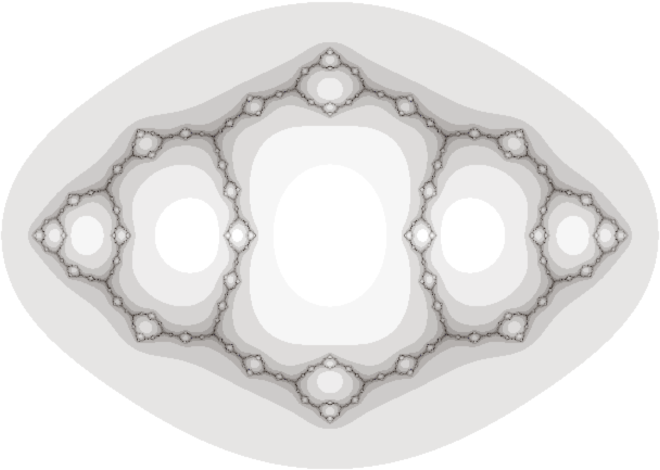
On $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, consider $f(z) = 1 - \dfrac{1}{z^2}$.

Cycle including critical points:



Near this cycle, $f^{\circ 3}$ looks like $z \mapsto z^4$. So nearby points are attracted.

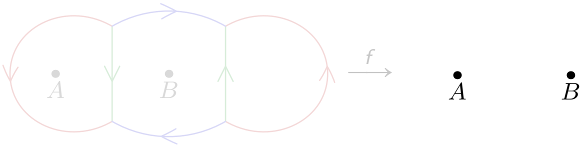
Rational maps: Julia set



Rational maps: Topological representation

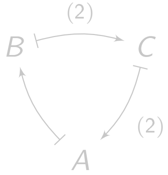
To draw a picture of a rational map f, with finite *post-critical set* P:

- Pick a spine Γ for $S^2 \backslash P$
- Draw $f^{-1}(\Gamma) \subset S^2 \backslash P$, a graph that covers Γ



(Point C at infinity)

This determines how f acts on P:



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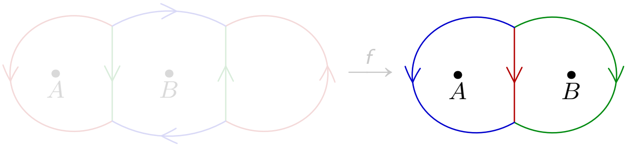

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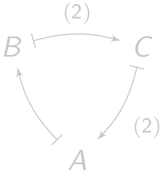

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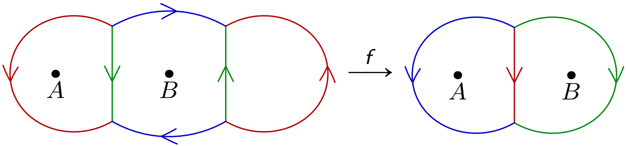

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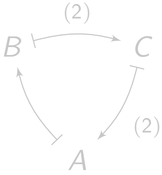

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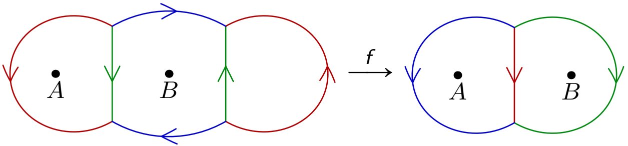

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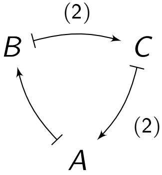
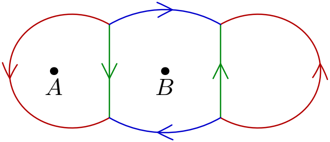


Problem statement

When is a branched self-cover equivalent to a post-critically finite (pcf) rational map?



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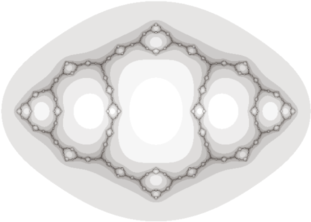

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Old answer: Iff there is no obstruction, certain collections of curves (W. Thurston, 1982)

Assumption

Branched self-covers have a branch point in each cycle in P. (Corresponding pcf maps are hyperbolic.)

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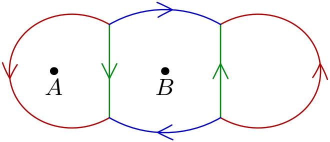

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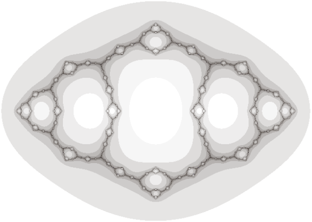

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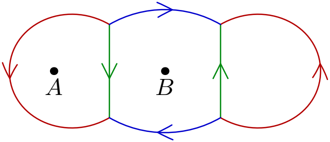

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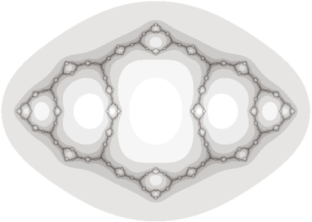

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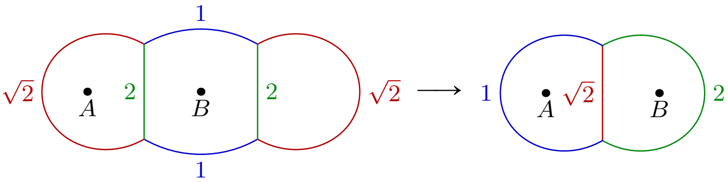
Answer

When is a branched self-cover f (post-critical set P) equivalent to a post-critically finite (pcf) rational map?

New answer: If $\exists$ *elastic spine* Γ for $S^2 \setminus P$ so that $f^{-1}(\Gamma)$ is *strictly looser* than Γ, then f is equivalent to a pcf map.



For converse direction, may need to iterate:
f rational $\Rightarrow f^{-n}(\Gamma)$ is evenutally looser than Γ.

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