# Detecting rational maps using elastic graphs

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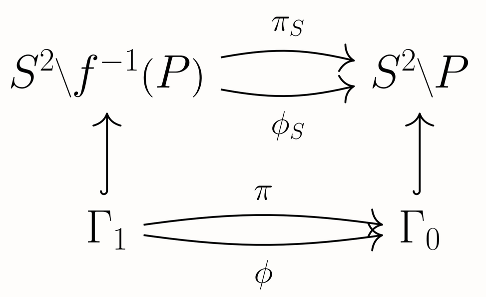
## Spines for branched self-covers

Given a branched self-cover of the sphere

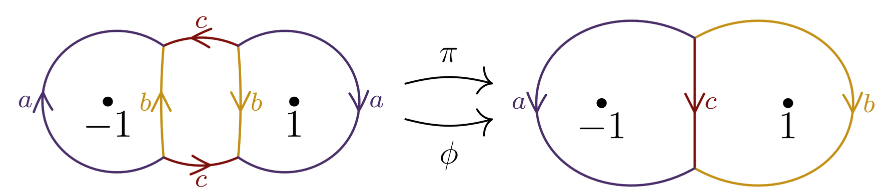
$$f\colon (S^2, P) \to (S^2, P).$$

This gives a *virtual endomorphism* or *topological automaton* of $S^2 \setminus P$ or of a spine for it




with $\pi_S$ a covering map (restriction of $f$), $\phi_S$ the natural inclusion map, and maps $\pi$ and $\phi$ on graphs commuting up to homotopy.

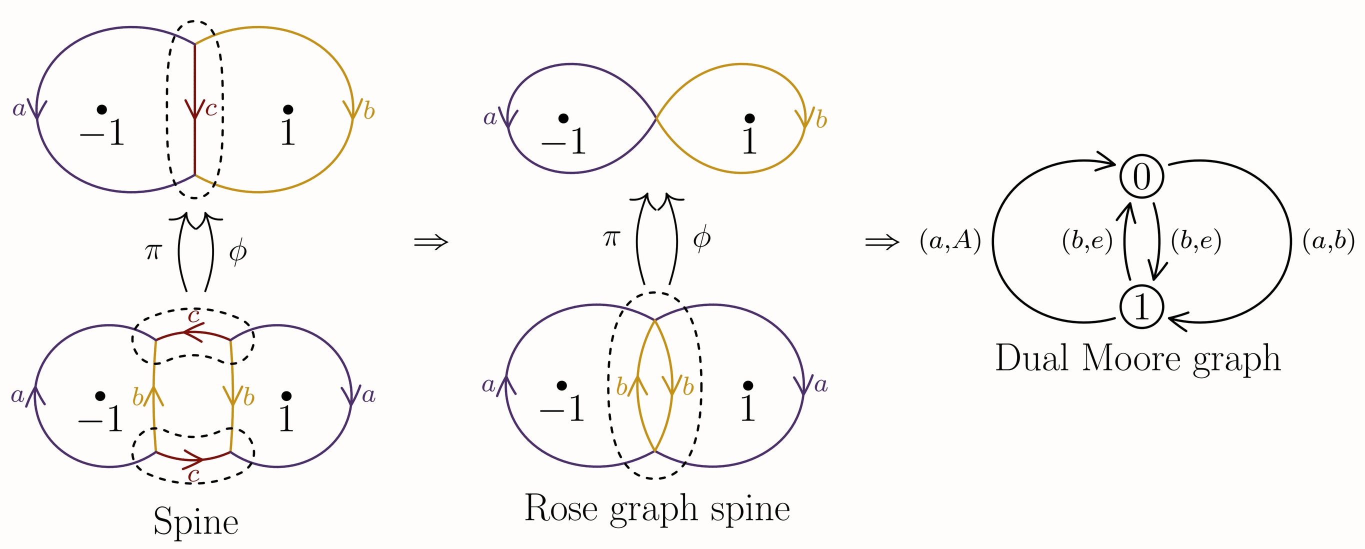
**Example: $f(z) = (1 + z^2)/(1 - z^2)$**



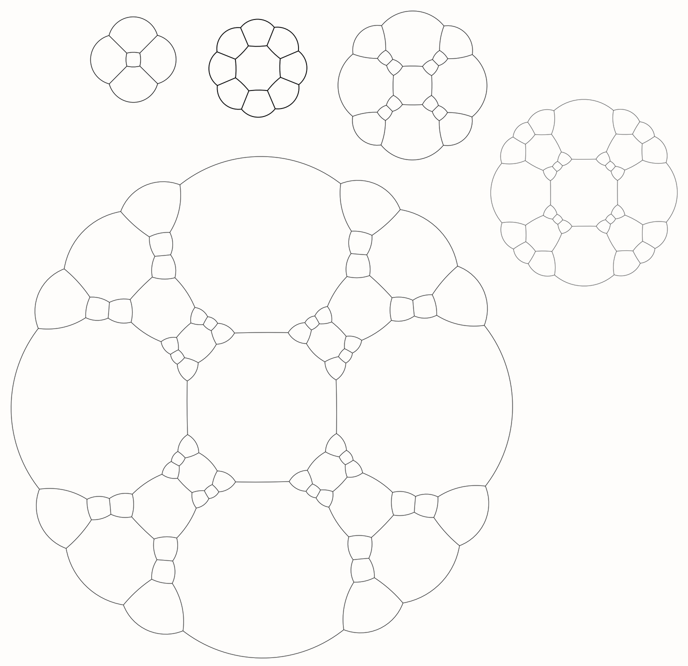

Critical portrait:

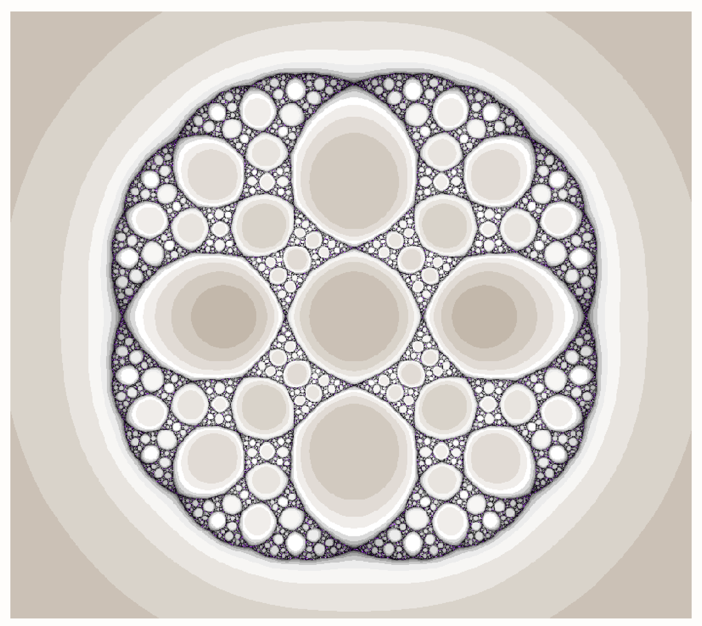
$$0 \xrightarrow{(2)} 1 \longrightarrow \infty \underset{}{\overset{(2)}{\rightleftarrows}} -1$$

## Spines to automata




## Iteration





## Main theorem

Let $f\colon (S^2, P) \circlearrowleft$ be a branched self-cover with at least one branch point in each cycle in $P$, and let $\pi, \phi\colon \Gamma_1 \to \Gamma_0$ be a corresponding virtual endomorphism. Then the following are equivalent:

- $f$ equivalent to a rational map
- for some metric on $\Gamma_0$ and some $n > 0$, $\phi_n\colon \Gamma_n \to \Gamma_0$ is loosening
- for every metric on $\Gamma_0$ and every $n \gg 0$, $\phi_n\colon \Gamma_n \to \Gamma_0$ is loosening

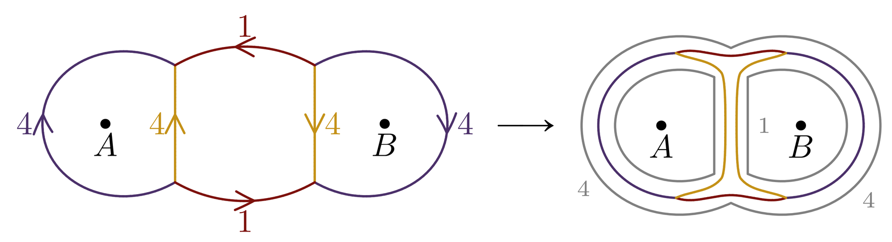
This is a positive criterion, unlike W. Thurston's annular obstruction

**Definition: Loosening**

If $\Gamma_1$, $\Gamma_0$ are metric graphs, a Lipshitz map $\phi\colon \Gamma_1 \to \Gamma_0$ is *loosening* if, for almost every $y \in \Gamma_0$,

$$\sum_{x \in \phi^{-1}(y)} |\phi'(x)| < 1.$$

**Example: $f(z) = (1 + z^2)/(1 - z^2)$**




## Graph energies

$$\text{Union of circles } C \xrightarrow{c} \text{Elastic graph } \Gamma_1 \xrightarrow{\phi} \text{Elastic graph } \Gamma_2 \xrightarrow{f} \text{Length graph } K$$

- $\mathrm{Emb}(\phi) = \sup_{y \in \Gamma_2} \sum_{x \in \phi^{-1}(y)} |\phi'(x)|$
- $\mathrm{Dir}(f) = \int_{x \in \Gamma_2} |f'(x)|^2 \, dx$
- $\mathrm{EL}(c) = \int_{y \in \Gamma_1} n_c(y)^2 \, dy$
- $\mathrm{SF}_{\mathrm{Dir}}[\phi] = \sup_{K,f} \dfrac{\mathrm{Dir}[f \circ \phi]}{\mathrm{Dir}[f]}$
- $\mathrm{SF}_{\mathrm{EL}}[\phi] = \sup_{C,c} \dfrac{\mathrm{EL}[\phi \circ c]}{\mathrm{EL}[c]}$

**Theorem**

$$\mathrm{Emb}[\phi] = \mathrm{SF}_{\mathrm{Dir}}[\phi] = \mathrm{SF}_{\mathrm{Emb}}[\phi].$$

## Surface energies

$$\text{Union of circles } C \xrightarrow{c} \text{Conformal surface } \Sigma_1 \xrightarrow{\phi} \text{Conformal surface } \Sigma_2 \xrightarrow{f} \text{Length graph } K$$

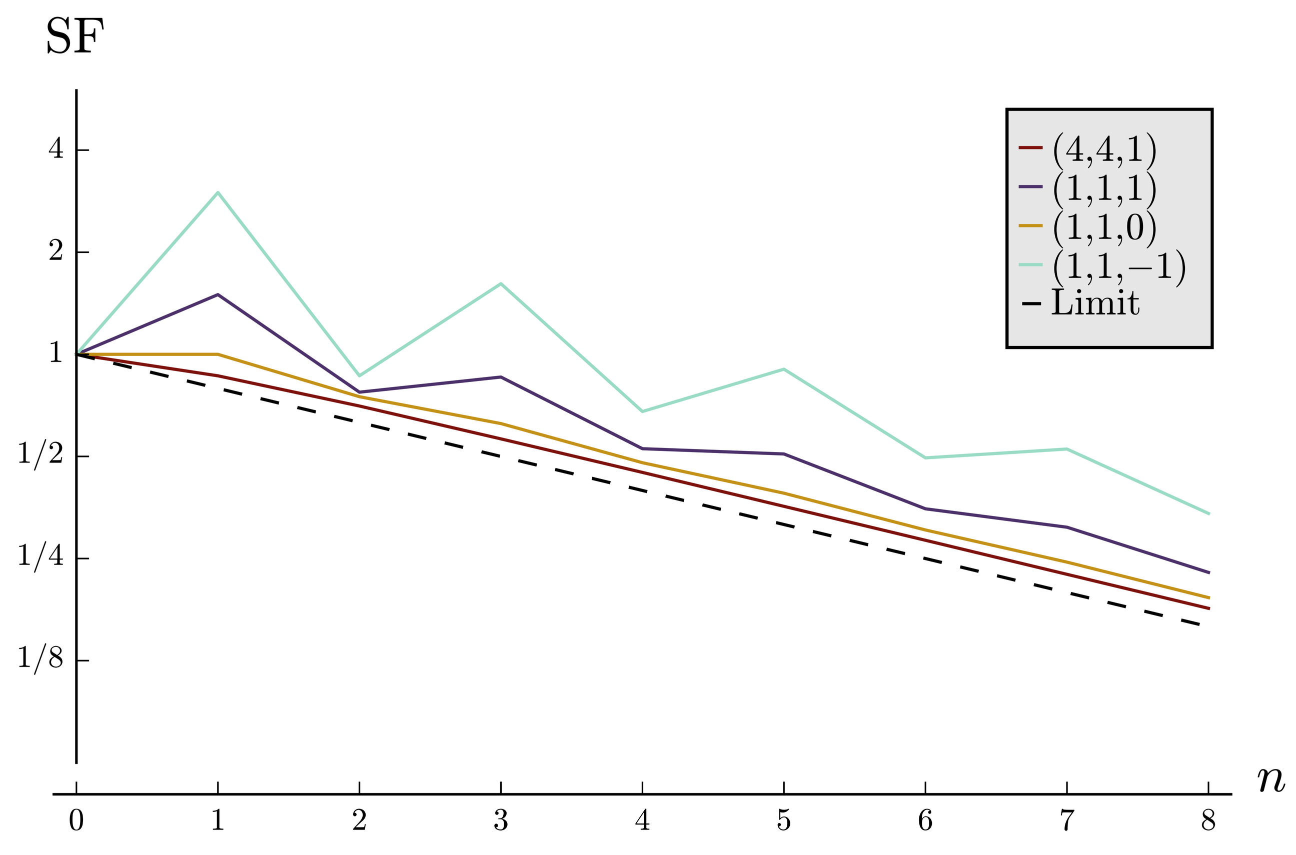
- *Dirichlet energy* $\mathrm{Dir}[f]$ of $f\colon \Sigma_2 \to K$
- *Extremal length* $\mathrm{EL}[c]$ of $c\colon C \to \Sigma_1$
- $\mathrm{SF}[\phi] = \sup_{C,c} \dfrac{\mathrm{EL}[\phi \circ c]}{\mathrm{EL}[c]}$.

**Theorem**

$$(1 - \varepsilon)\,\mathrm{SF}_{\mathrm{Graph}}[\phi] \leqslant \mathrm{SF}_{\mathrm{Surf}}[N_t\phi] \leqslant (1 + \varepsilon)\,\mathrm{SF}_{\mathrm{Graph}}[\phi],$$

where $t \ll 1$ and $\varepsilon$ depends only on local geometry.

## Experimental output




Logarithmic plot of SF, iterating running example with varying elastic lengths and topology. Dashed line shows limiting behaviour: $\mathrm{SF}[\phi_n] \geqslant 2^{-n/3}$.

## Asymptotic stretch factors

Give $(\pi, \phi)$ a virtual endomorphism of graphs $\Gamma_1, \Gamma_0$ or surfaces $S_1, S_0$.

$$\overline{\mathrm{SF}}[\phi] := \lim_{n \to \infty} \sqrt[n]{\mathrm{SF}[\phi_n]}.$$

**Lemma: Independence of starting data**

$\mathrm{SF}[\phi]$ is independent of elastic weights on $\Gamma_0$ or conformal structure on $S_0$.

**Proof**

$$\mathrm{SF}[\phi \circ \psi] \leqslant \mathrm{SF}[\phi]\,\mathrm{SF}[\psi]$$
$$\mathrm{SF}[\Gamma_n' \to \Gamma_0'] \leqslant \mathrm{SF}[\Gamma_n' \to \Gamma_n]\,\mathrm{SF}[\Gamma_n \to \Gamma_0]\,\mathrm{SF}[\Gamma_0 \to \Gamma_0']$$
$$\leqslant K_1\,\mathrm{SF}[\Gamma_n \to \Gamma_0] K_2.$$

## Proof sketch

$f$ is equivalent to a rational map iff there is a virtual endomorphism of conformal surfaces $\pi, \phi\colon S_1 \to S_0$, where $\pi$ is a covering and $\phi$ is a strict conformal embedding (Cui–Peng–Lei).

For $\phi\colon S_1 \hookrightarrow S_0$, $\mathrm{SF}_{\mathrm{Surf}}(\phi) < 1$ iff there is a strict conformal embedding in $[\phi]$ (Kahn–Pilgrim–T.)

Thus $f$ is equivalent to a rational map iff $\overline{\mathrm{SF}}[\phi] < 1$.

If $\overline{\mathrm{SF}}[\phi] < 1$, must have $\mathrm{SF}[\phi_n] < 1$ for some $n$.