

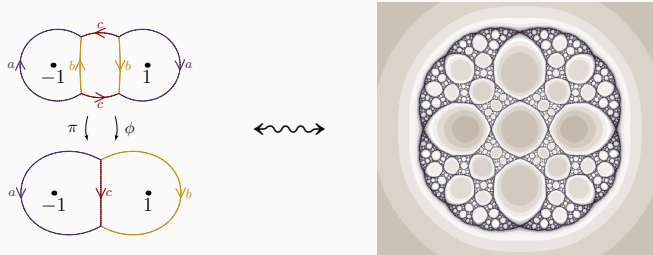
# Detecting rational maps using elastic graphs

Dylan Thurston

*In Memoriam: William P. Thurston, 1946–2012*

Portions joint with J. Kahn and K. Pilgrim

<http://pages.iu.edu/~dpthurst/writing/DetectRational.pdf>



August 18, 2015

# Spines for branched self covers

Branched self-cover of sphere

$$f: (S^2, P) \rightarrow (S^2, P).$$

Gives *virtual endomorphism* or *topological automaton* of a spine:

$$\begin{array}{ccc} S^2 \setminus f^{-1}(P) & \xrightleftharpoons[\phi_S]{\pi_S} & S^2 \setminus P \\ \uparrow & & \uparrow \\ \Gamma_1 & \xrightleftharpoons[\phi]{\pi} & \Gamma_0 \end{array}$$

with  $\pi_S$  a covering map (restriction of  $f$ ),  $\phi_S$  the inclusion map, and maps  $\pi$  and  $\phi$  on graphs commuting up to homotopy.

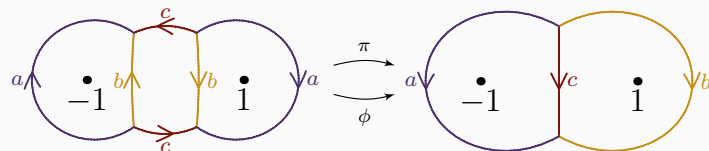
*Spine* for  $\Sigma$ : graph that fills  $\Sigma$  (complement: punctured disks)

Example:  $f(z) = (1 + z^2)/(1 - z^2)$

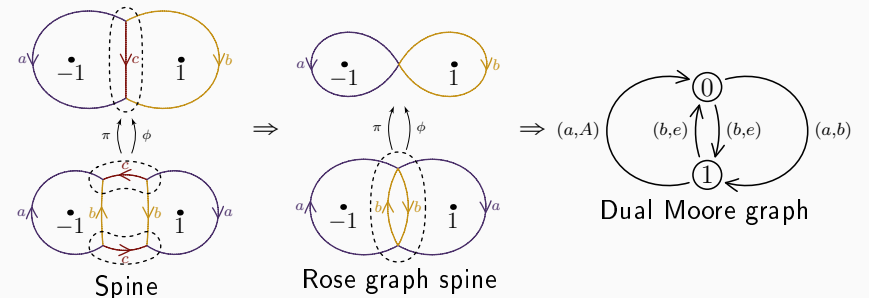
Critical portrait:

$$0 \xrightarrow{(2)} 1 \longrightarrow \infty \xrightleftharpoons{(2)} -1$$

Set  $P = \{-1, 1, \infty\}$

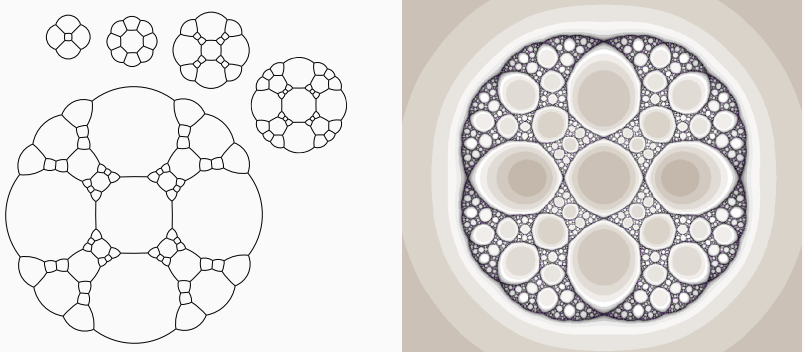
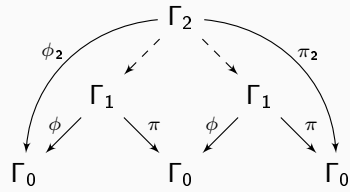


# Spines to automata



- Collapse maximal tree in  $\Gamma_0$  and lift in  $\Gamma_1$  to get  $R_0, R_1$
- Homotop  $\phi$  so vertices of  $R_1$  map to vertex of  $R_0$
- Label edges by  $\pi_1(R_0)$

# Iteration



# Main theorem

## Theorem

Let  $f: (S^2, P) \looprightarrow$  be a branched self-cover with at least one branch point in each cycle in  $P$ , and let  $\pi, \phi: \Gamma_1 \rightarrow \Gamma_0$  be a corresponding virtual endomorphism.

Then the following are equivalent:

- $f$  equivalent to a rational map
- for some metric on  $\Gamma_0$  and some  $n > 0$ ,  $\phi_n: \Gamma_n \rightarrow \Gamma_0$  is loosening
- for every metric on  $\Gamma_0$  and every  $n \gg 0$ ,  $\phi_n: \Gamma_n \rightarrow \Gamma_0$  is loosening

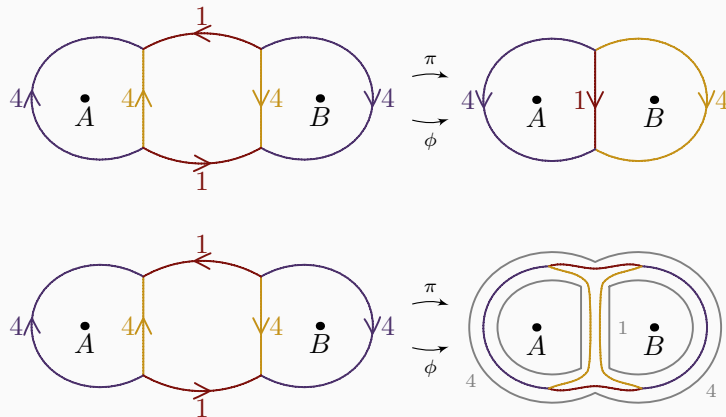
## Definition

If  $\Gamma_1, \Gamma_0$  are metric graphs, a Lipschitz map  $\phi: \Gamma_1 \rightarrow \Gamma_0$  is *loosening* if, for almost every  $y \in \Gamma_0$ ,

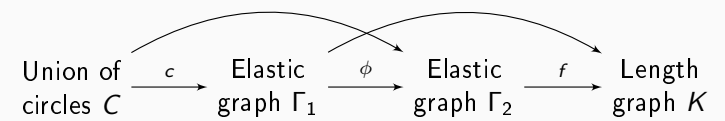
$$\sum_{x \in \phi^{-1}(y)} |\phi'(x)| < 1.$$

In particular,  $\phi$  is 1-Lipschitz.

Example:  $f(z) = (1 + z^2)/(1 - z^2)$



# Energies for graph maps



- Elastic graph: graph with *spring constant* on each edge
- Length graph: graph with *length* on each edge
- $\text{Emb}(\phi) = \sup_{y \in \Gamma_2} \sum_{x \in \phi^{-1}(y)} |\phi'(x)|$
- $\text{Dir}(f) = \int_{x \in \Gamma_2} |f'(x)|^2 dx$
- $\text{EL}(c) = \int_{y \in \Gamma_1} n_c(y)^2 dy$

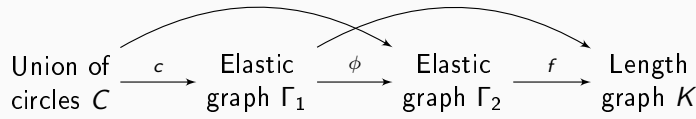
Submultiplicative...

- $\text{Dir}(f \circ \phi) \leq \text{Emb}(\phi) \text{Dir}(f)$
- $\text{EL}(\phi \circ c) \leq \text{EL}(c) \text{Emb}(\phi)$

... minimizing over homotopy class:

- $\text{Dir}[f \circ \phi] \leq \text{Emb}[\phi] \text{Dir}[f]$
- $\text{EL}[\phi \circ c] \leq \text{EL}[c] \text{Emb}[\phi]$

## Stretch factors for graph maps



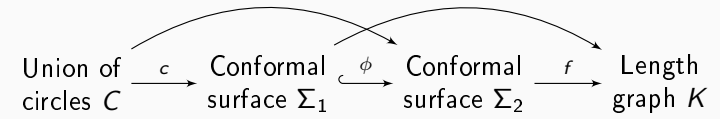
- $\text{Emb}(\phi) = \sup_{y \in \Gamma_2} \sum_{x \in \phi^{-1}(y)} |\phi'(x)|$
- $\text{Dir}(f) = \int_{x \in \Gamma_2} |f'(x)|^2 dx$
- $\text{EL}(f) = \int_{y \in \Gamma_1} n_c(y)^2 dy$
- $\text{SF}_{\text{Dir}}[\phi] = \sup_{K,f} \frac{\text{Dir}[f \circ \phi]}{\text{Dir}[f]}$
- $\text{SF}_{\text{EL}}[\phi] = \sup_{C,c} \frac{\text{EL}[\phi \circ c]}{\text{EL}[c]}$

### Theorem

For any  $\phi: \Gamma_1 \rightarrow \Gamma_2$ ,

$$\text{Emb}[\phi] = \text{SF}_{\text{Dir}}[\phi] = \text{SF}_{\text{EL}}[\phi].$$

## Energies and stretch factors for surface maps



- For  $f: \Sigma_2 \rightarrow K$ , have *Dirichlet energy*.
- For  $c: C \rightarrow \Sigma_1$  an embedded simple closed multi-curve, have *extremal length*  $\text{EL}[c]$ .
- For  $\phi: \Sigma_1 \hookrightarrow \Sigma_2$  a topological embedding,  $\text{SF}[\phi] = \sup_{C,c} \frac{\text{EL}[\phi \circ c]}{\text{EL}[c]}$ .

### Theorem (Kahn, Pilgrim, T)

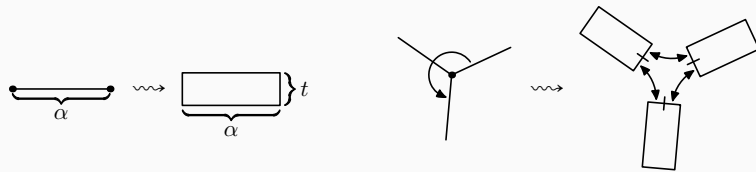
$\text{SF}[\phi] \leq 1 \Leftrightarrow \phi$  homotopic to conformal embedding

$\text{SF}[\phi] < 1 \Leftrightarrow \phi$  homotopic to conformal embedding with some space

If  $\text{SF}[\phi] > 1$ , then  $\text{SF}[\phi]$  is the minimal quasi-conformal constant in  $[\phi]$ .  
General interpretation??

## Relating graphs and surfaces

Elastic ribbon graph  $\Gamma \rightsquigarrow$  Conformal surface  $N_t \Gamma$



### Proposition

For  $\Gamma$  an elastic ribbon graph,  $t \ll 1$ , and  $c$  a curve on  $\Gamma$

$$\text{EL}_{\Gamma}[c] \leq t \cdot \text{EL}_{N_t \Gamma}[c] \leq (1 + \varepsilon) \text{EL}_{\Gamma}[c],$$

where  $\varepsilon$  depends only on the local geometry of  $\Gamma$ .

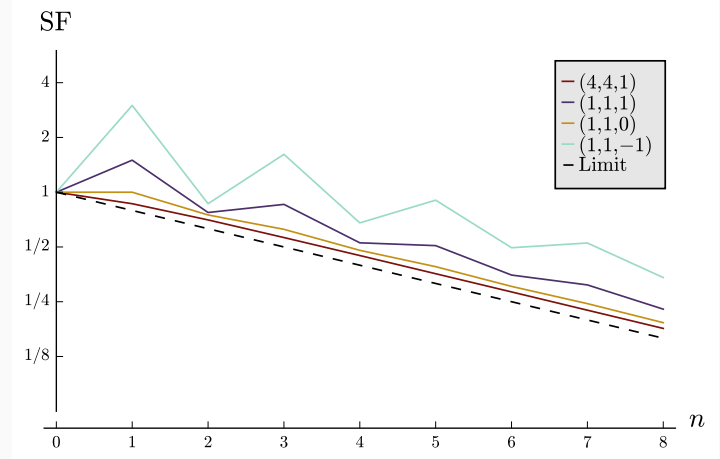
### Corollary

For  $\phi: \Gamma_1 \rightarrow \Gamma_2$  a suitable map between elastic ribbon graphs and  $t \ll 1$ ,

$$(1 - \varepsilon) \text{SF}[\phi] \leq \text{SF}[N_t \phi] \leq (1 + \varepsilon) \text{SF}[\phi]$$

where  $\varepsilon$  depends only on the local geometry of  $\Gamma_1$  and  $\Gamma_2$ .

## Experimental data



Logarithmic plot of SF, iterating running example with varying elastic lengths and topology.

Dashed line shows limiting behaviour:  $\text{SF}[\phi_n] \geq 2^{-n/3}$  in this case.

## Key to main theorem: Asymptotics of stretch factors

### Definition

For  $\pi, \phi: \Gamma_1 \rightarrow \Gamma_0$  a virtual endomorphism of elastic graphs, the *asymptotic stretch factor* is

$$\overline{\text{SF}}[\phi] = \lim_{n \rightarrow \infty} \sqrt[n]{\text{SF}[\phi_n]}.$$

### Lemma

$\overline{\text{SF}}[\phi]$  is independent of elastic weights on  $\Gamma_0$ .

### Proof.

$$\begin{aligned} \text{SF}[\phi \circ \psi] &\leq \text{SF}[\phi] \text{SF}[\psi] \\ \text{SF}[\Gamma'_n \rightarrow \Gamma'_0] &\leq \text{SF}[\Gamma'_n \rightarrow \Gamma_n] \text{SF}[\Gamma_n \rightarrow \Gamma_0] \text{SF}[\Gamma_0 \rightarrow \Gamma'_0] \\ &= K_1 \text{SF}[\Gamma_n \rightarrow \Gamma_0] K_2. \end{aligned} \quad \square$$

## Asymptotic stretch factor for surfaces

### Definition

For  $\pi, \phi: S_1 \rightarrow S_0$  a virtual endomorphism of surfaces, the *asymptotic stretch factor* is

$$\overline{\text{SF}}_{\text{Surf}}[\phi] = \lim_{n \rightarrow \infty} \sqrt[n]{\text{SF}[\phi_n]}.$$

### Lemma

$\overline{\text{SF}}_{\text{Surf}}[\phi]$  is independent of conformal structure on  $S_0$ .

### Lemma

$$\overline{\text{SF}}_{\text{Surf}}[\phi] = \overline{\text{SF}}_{\text{Graph}}[\phi].$$

### Proof.

Use same technique as above, combined with result on approximating SF for  $\Gamma$  and  $N_t \Gamma$ . □

## Completing the proof

### Theorem

Let  $f: (S^2, P) \looparrowright$  be branched self-cover, at least one branch point in each cycle in  $P$ . Then  $f$  equivalent to a rational map  $\Leftrightarrow$  exists surface  $\Sigma_0 \subset S^2 \setminus P$  so that

$$\Sigma_1 \begin{array}{c} \xrightarrow{\pi} \\ \xrightarrow{\phi} \end{array} \Sigma_0$$

with  $\phi$  a conformal embedding.

### Proof.

Folklore, using quasi-conformal surgery; written by Cui–Peng–Tan. □

### Corollary

$\overline{\text{SF}}_{\text{Surf}}[\phi] < 1$  iff  $(\pi, \phi)$  is equivalent to a rational map

### Proof.

Combine theorem above with characterization of surface embedding by SF. □

## Completing the proof, cont

### Corollary

$\overline{\text{SF}}_{\text{Surf}}[\phi] < 1$  iff  $(\pi, \phi)$  is equivalent to a rational map

### Theorem

Let  $f: (S^2, P) \looparrowright$  be a branched self-cover with at least one branch point in each cycle in  $P$ , and let  $\pi, \phi: \Gamma_1 \rightarrow \Gamma_0$  be a corresponding virtual endomorphism.

Then the following are equivalent:

- $f$  equivalent to a rational map
- for some metric on  $\Gamma_0$  and some  $n > 0$ ,  $\phi_n: \Gamma_n \rightarrow \Gamma_0$  is loosening
- for every metric on  $\Gamma_0$  and every  $n \gg 0$ ,  $\phi_n: \Gamma_n \rightarrow \Gamma_0$  is loosening

## Questions

- Census of rational maps? (Table of 464 rational maps vs. 1,701,935 prime knots)
- Apply criterion in concrete cases (e.g., matings)?
- Polynomial-sized certificates?
- Is  $\overline{SF}[\phi]$  always algebraic? How to compute it?
- New proof of W. Thurston's annular obstruction?
- Direct interpretation of  $SF[\phi]$  for  $\phi: S_1 \rightarrow S_2$  when  $SF[\phi] < 1$ ? (When  $SF[\phi] \geq 1$ , it equals the minimal quasi-conformal dilatation.)
- What happens for topological automata that do not come from rational maps? What does  $\overline{SF}[\phi] < 1$  mean in general?