

# Hyperbolic Volume and the Jones Polynomial

## The Conjectures

Recall that the <sup>unframed,</sup> colored Jones polynomial  $J_\alpha(K)$  depends on an integer,  $\alpha$ , essentially the number of strands, and is a polynomial in a parameter  $q^{1/2}$ . If we take  $q$  to be a Laurent

root of unity,  $q^N = 1$ , then we have various symmetries; in particular,  $J_{2N-\alpha}(K) = J_\alpha(K)$ , so  $J_N(K) = 0$ . In fact,  $J_\alpha(K)$  is always divisible by  $(q^{\alpha/2} - q^{-\alpha/2})$  (for generic  $q$ ). Define a new polynomial  $J'_\alpha(K) = J_\alpha(K) / (q^{\alpha/2} - q^{-\alpha/2})$  (essentially equivalent to breaking one strand of the knot to get a 1-strand tangle.)

We then have the following

### Conjecture (Kashaev-Murakami-Murakami)

For knots  $K$  with a hyperbolic complement, ~~the volume is~~

$$\text{Vol}(K) = 2\pi \lim_{N \rightarrow \infty} \frac{\log |J'_N(K)|_{q=e^{2\pi i/N}}}{N}$$

This provides a striking link between the Jones polynomial and classical geometry and topology, in a rather unexpected direction. In some sense, this is on "the other end" from Vassiliev invariants; for instance, if you look at Rozansky's expressions for the colored Jones polynomial as a Melvin-Morton type expansion, each term in the sum goes to zero in this limit.

There are some stranger versions of this conjecture; for instance, we have:

### Conjecture (Murakami-Murakami) For any knot $K$ ,

$$\|K\| = \text{const} \cdot \lim_{N \rightarrow \infty} \frac{\log |J'_N(K)|_{q=e^{2\pi i/N}}}{N}$$

where  $\|K\|$  is the Gromov norm, mentioned by Baileu ~~is equal to~~ (equal to a constant times the sum of volumes of the hyperbolic pieces for geometric manifolds).

I'm not sure I believe this version, because there is a competing

Conjecture (Kashaev's version) For hyperbolic links  $L$ ,

$$\text{vol}(L) = \lim_{N \rightarrow \infty} \frac{\log |J_{N,N,-N}(L) / q^{N-1}|_{q=e^{2\pi i/N}}}{N}$$

~~For~~ This is competing, since for split links the RHS vanishes, while the Gromov norm need not be zero.

There are some further extensions one could ask for:

Question

~~Does  $\lim_{N \rightarrow \infty} \frac{\log |J_{N,N,-N}(L)|}{N}$  approach the Chern-Simons invariant?~~

If you remove the absolute value signs, does the <sup>imaginary part of the</sup> limit approach the Chern-Simons invariant? (Note that there are some problems of definition here.)

Question

Does  $\lim_{N \rightarrow \infty} \frac{\log |J_{\alpha}(K)(e^{2\pi i/N})|}{N}$ ,  $\frac{\alpha}{N}$  fixed

approach the volume of the hyperbolic structure with cone angle  $q^{\alpha}$ ? [I'll explain this more later. This would be most interesting for imaginary  $\alpha$ .]

Question For a hyperbolic 3-manifold  $M$ , does

$$\text{vol}(M) = 2\pi \lim_{N \rightarrow \infty} \frac{\log |RT_{e^{2\pi i/N}}(M)|}{N} ?$$

Answer No: The RT invariants grow polynomially in  $N$ . DPT Jun 20, 2000

$[RT_q$  is the Reshetikhin-Turaev invariant of  $M$ . This would be surprising, since the Reshetikhin-Turaev invariants are supposed to only see  $SU(2)$  connections, not  $SL(2, \mathbb{C})$ ; but this is supposed to be true for the Jones polynomial, too, and as far as I know the only calculations have been done for non-hyperbolic manifolds.

Note the earlier conjectures are essentially the case when  $M$  has at least one loop.]

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Today I'd like to explain why I believe these conjectures. In particular, there seems to be a rather precise correspondence between ideal triangulations and the formulas one writes down for the colored Jones polynomial in which each quantum factorial corresponds to an ideal tetrahedron.

Note that I don't believe the conjectures are so hard that they'll remain conjectures very long, given their importance. But this will only open up a large territory to explore. You can think of this talk as "Coming Attractions."

### Notation + Useful Facts

$q$  is an  $N$ th root of unity,  $e^{2\pi i/N}$ ;  $q^{1/2} = e^{\pi i/N}$ .

$$[n!] = (1-q)(1-q^2) \cdots (1-q^n)$$

$$(n!) = (q^{1/2} - q^{-1/2})(q - q^{-1}) \cdots (q^{n/2} - q^{-n/2})$$

N.B.: no factor  $(1-q)^n$ , as would be standard.

$$[N-1!] = N \quad (\text{pt. } (x-1)(x-q) \cdots (x-q^{N-1}) = x^N - 1; \text{ divide both sides by } x-1 \text{ + evaluate at } x=1)$$

$$[n!]^* = (1-q^{-1})(1-q^{-2}) \cdots (1-q^{-n}) = (-1)^n q^{-n(n+1)/2} [n!]$$

Exercises called "Extra Credit" are those I haven't done.



(Due to Thang Le + Tobias Ekholm)

(4)

Using canonical bases, Le found the following formula for the Figure 8 knot:

$$J'_\alpha(\mathcal{B}) = \sum_{k=0}^{\alpha-1} (1-q^{\alpha-1})(1-q^{\alpha-2}) \dots (1-q^{\alpha-k})(1-q^{\alpha+1})(1-q^{\alpha+2}) \dots (1-q^{\alpha+k}) q^{-\alpha k}$$

$$= \sum_{k=0}^{\alpha-1} \frac{q^{-\alpha k} [\alpha+k!]}{[\alpha-k-1!](q-q^\alpha)} = \sum_{k=0}^{\alpha-1} \frac{(q+k!)}{(\alpha-k-1!)(q^{\alpha/2}-q^{\alpha/2})}$$

Specializing to  $\alpha=N$ , we find

$$J'_N(\mathcal{B}) = \sum_{k=0}^{N-1} (k!)^2$$

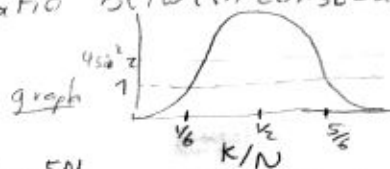
Each term in this sum is positive, so we have

$$\max_k (k!)^2 \leq J'_N(\mathcal{B}) \leq N \max_k (k!)^2$$

$$\lim_{N \rightarrow \infty} \frac{\log |J'_N(\mathcal{B})|}{N} = \lim_{N \rightarrow \infty} \max_{0 \leq k < N} (k!)^2$$

To find this maximum, consider the ratio between consecutive terms!

$$\frac{(k!)}{(k-1)!} = (q^{k/2} - q^{k/2})^2 = 4 \sin^2\left(\frac{\pi k}{N}\right)$$



This is  $\leq 1$  for  $0 < k \leq \frac{N}{6}$ ,  $\geq 1$  for  $\frac{N}{6} \leq k \leq \frac{5N}{6}$ , and  $\leq 1$  for  $k \geq \frac{5N}{6}$ .

Thus the product of these terms achieves the maximum at  $k = \frac{5N}{6}$  or  $q^k = e^{-\pi/3}$ .

What is this maximum? Take the log:

$$\log \left( \left( \frac{5N}{6}! \right)^2 \right) = 2 \sum_{\ell=1}^{5N/6} \log \left( 2 \sin \frac{\pi \ell}{N} \right) \stackrel{(*)}{\sim} 2 \frac{N}{\pi} \int_{\theta=0}^{5\pi/6} \log(2 \sin \theta) d\theta = -\frac{2N}{\pi} \Lambda\left(\frac{5\pi}{6}\right)$$

using identities from Boileau's lecture:

$$\Lambda\left(\frac{5\pi}{6}\right) = -\Lambda\left(\frac{\pi}{6}\right) ; \quad \frac{1}{2} \Lambda\left(\frac{\pi}{3}\right) = \Lambda\left(\frac{\pi}{6}\right) + \Lambda\left(\frac{2\pi}{3}\right) \Rightarrow \Lambda\left(\frac{\pi}{6}\right) = \frac{3}{2} \Lambda\left(\frac{\pi}{3}\right)$$

So

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J'_N(\mathcal{B})|}{N} = 2\pi \cdot \frac{2}{\pi} \cdot \frac{3}{2} \Lambda\left(\frac{\pi}{3}\right) = 6 \Lambda\left(\frac{\pi}{3}\right), \text{ as desired.}$$

Note: at (\*) we approximated a sum by an integral. This can be justified using, for instance, the Euler-Maclaurin summation formula. See [Kir, section 2.5] for details and calculation of lower order terms.

What have we learned from this example? <sup>Three</sup> ~~Two~~ key points:

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- (a) Terms in the sum with  $\frac{k}{N}$  some fixed value are growing exponentially. Since we're looking for the overall exponential growth, it suffices to look for the term that grows the fastest.
- (b) To find the maximum term, set the ratio of adjacent terms equal to 1.
- (c) At the maximum, each quantum factorial turns into a  $\Lambda$  function (or, alternatively, a dilogarithm), corresponding to the volume of a tetrahedron. (Boileau used 3  $\Lambda$ -functions per tetrahedron in his talk, but I'll show later how to get by with just 1.)



Emboldened by our success with the first hyperbolic knot, let's try the next one.

In [Kash<sup>9</sup>], Kashaev gives the following formula

for the knot  $S_2$ :

$$\langle S_2 \rangle = N^2 \sum_{k+l=N-1} \frac{(-1)^k q^{2l(l+1) + k(k-1)/2 + lk}}{[l!]^2 [k!]} =: N^2 \sum_{k+l=N-1} s(k, l)$$

[N.B: This is Kashaev's invariant, which is only defined up to a root of unity, and which according to [MM] agrees with  $J_N$ .]

Let's again try to find maxima/minima (assuming they're not near the boundary):

$$(a) \quad s(k, l) = \frac{q^{2l+k}}{(1-q^2)^2} s(k, l-1)$$

$$(b) \quad s(k, l) = \frac{-q^{k+l}}{(1-q^k)} s(k-1, l)$$

Setting both ratios to 1 (as at a local maximum), we find

$$(a) \Rightarrow q^k = \frac{(1-q^{2l})^2}{q^{2l}}$$

$$(b) \Rightarrow q^l = -\frac{1-q^k}{q^k} = -q^{-k} + 1 = -\frac{q^{2l}}{(1-q^2)^2} + 1 = \frac{+1 - 2q^2}{(1-q^2)^2}$$

~~$$q^{3l} - 2q^{2l} + 3q^l - 1 = 0$$~~

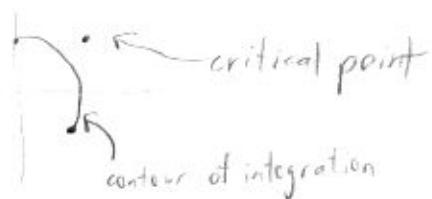
which is a certain cubic equation for  $l$  which we solve to get the critical values. However, something strange is going on, since these critical values are not on the unit circle, meaning that they won't appear in the sum!

## Digression: Stationary Phase

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To see what stationary points off the contour of ~~the~~ summation mean, imagine that we had an integral rather than a sum.

Imagine that you do the integral <sup>of the exponential of  $f$</sup>  over some curve in the complex plane, and the critical point <sup>of  $f$</sup>  is somewhere off the curve.



Now draw the level curves of the real part of  $f$ . In the good case, they'll look like this:



Then we can deform the contour of integration to the dotted curve without changing the integral; and now it's clear that the region near the critical point will dominate the asymptotics.

In our situation, we have a sum rather than an integral, and we don't know enough about the function in question to do Euler-Maclauren summation with confidence; also, we set the "ratio of two terms" equal to 1, not the derivative. According to Lev, these two exactly cancel, and we really are finding the appropriate stationary points here. I have not verified this.

More serious is that we don't know if whatever stationary points we find are relevant; i.e., if, deforming the contour to decrease the real part of  $f$ , we hit our chosen stationary point.

In my opinion, this is the major open point to a proof of the conjectures.

Let's at least be precise about the answer we expect, assuming the stationary phase approximation works. Let  $z_0$  be the appropriate solution to the cubic equation

$$z_0^3 - 2z_0^2 - z_0 + 1 = 0 \quad ("=q^e")$$

$$w_0 = \frac{(1-z_0)^2}{z_0^2} \quad ("=q^k")$$

Then the term in the sum,

$$N^2 \frac{q^{(l-1) + k(k+1)/2 + lk} (-1)^k}{[l!]^2 [k!]}$$

would grow exponentially like

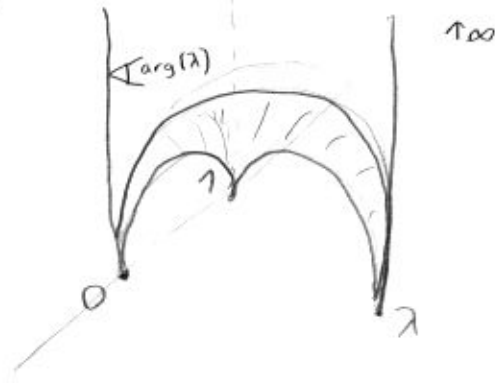
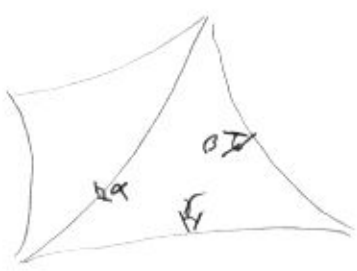
$$\exp\left[\frac{N}{2\pi i} \left( 2\text{Li}_2(z_0) + \text{Li}_2(w_0) + (\log z_0)^2 + \frac{1}{2}(\log w_0)^2 + \log z_0 \log w_0 \right)\right]$$

Some More Hyperbolic Geometry

Here  $\text{Li}_2(z) = -\int_1^z \log(1-z) \frac{dz}{z}$  is Roger's Dilogarithm.

Formulas like the one above (with dilogarithms evaluated at algebraic numbers) occur typically in the study of hyperbolic volumes. To see why, let's take another look at ideal tetrahedra and triangulations.

Boileau used ~~two~~ real numbers to parametrize an ideal tetrahedron ( $\alpha, \beta, \gamma$ , with  $\alpha + \beta + \gamma = \pi$ ).

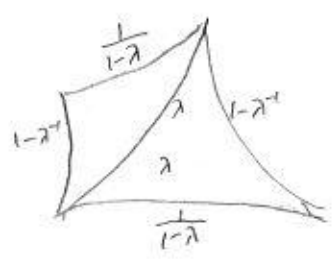


There is an alternate parametrization by a single complex number, the cross-ratio. One way of describing it is to <sup>use a hyperbolic isometry to</sup> put 3 of the 4 vertices at the points 0, 1, and infinity; then the remaining point will be at some point, lambda, which is the cross-ratio.  $\frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_2)(z_1 - z_3)}$

where  $z_0, z_1, z_2, z_3$  are the original points.

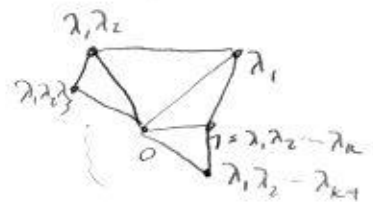
In fact, the two parametrizations are related. As indicated in the figure, a tetrahedron with cross-ratio lambda will also have a dihedral angle of arg lambda. The symmetries of the cross-ratio mean that it is naturally associated with an edge (up to  $\lambda \leftrightarrow \frac{1}{\lambda}$ )

If we do this for all the edges, we see



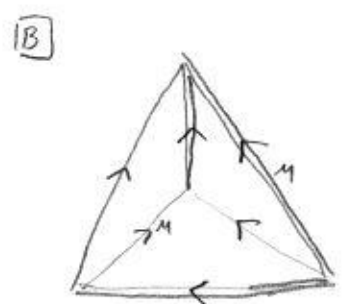
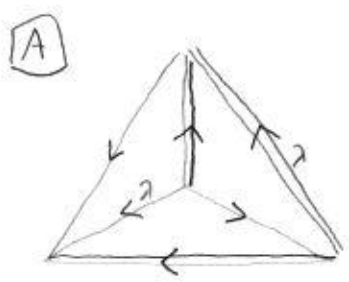
Note that  $\lambda \cdot (1-\lambda)^{-1} \cdot \frac{1}{1-\lambda} = -1$ , agreeing with  $\alpha + \beta + \gamma = \pi$ . We can think of the cross-ratios as complexified dihedral angles. ~~(The absolute value measures the translation distance along the edge.)~~  
 (The absolute value measures the translation distance along the edge.)

Now imagine what happens if we glue tetrahedra (with parameters  $\lambda_1, \dots, \lambda_k$ ) along an edge. Put the two vertices of the edge in question at 0 and  $\infty$ , and put the first vertex of the first tetrahedron at 1. Then successive vertices will be at  $\lambda_1, \lambda_1 \lambda_2, \dots, \lambda_1 \lambda_2 \dots \lambda_k$ . But the last vertex must be the same as the first, so for consistency we must have  $\prod \lambda_i = 1$ .



This generalizes the condition on the dihedral angles that  $\sum \theta_i = 2\pi$ .

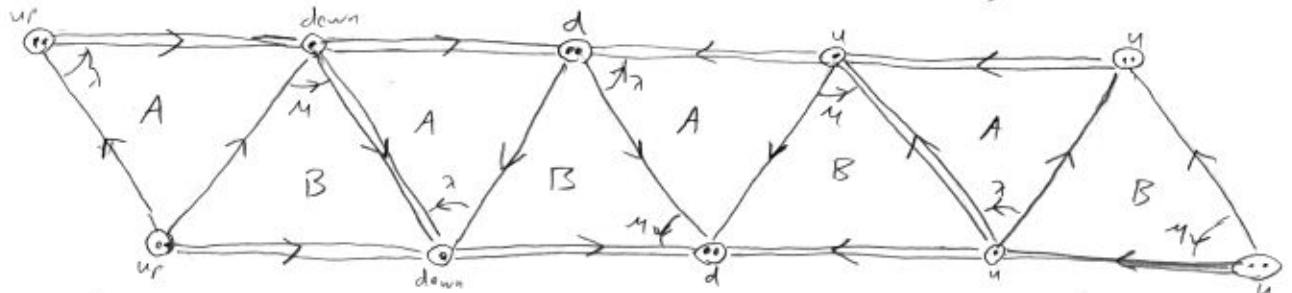
Let's see this in a particular example. Remember that the complement of  $\mathbb{G}$  could be obtained by gluing two tetrahedra like so:



(Note that the edge decorations determine the face gluings.)



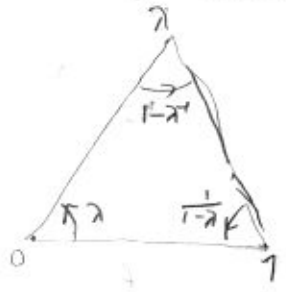
If we cut off a neighborhood of the vertices and look at how the resulting triangles are glued together, we get a picture like this: (top+bottom and left+right are glued)



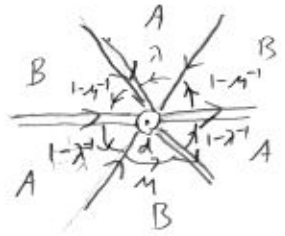
(The markings indicate the edges that come together at the vertex of the tetrahedra (on the vertices of the triangles above), as well as the marking on the opposite face.)

Exercise Verify this.

For plane triangles, the rules for labelling angles are simple: an angle is labelled by the complex ratio of the two adjoining sides. These ~~are~~ parameters are related like this:



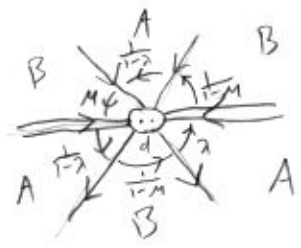
We can then read around the edge labelled  $\odot_{\text{down}}$ :



so, for geometric tetrahedra we must have

$$(*) \quad \lambda(1-\lambda)^2 \mu(1-\mu)^2 = \frac{(1-\lambda)^2}{\lambda} \frac{(1-\mu)^2}{\mu} = 1.$$

The vertex labelled  $\odot_{\text{up}}$  is the same edge in the simplicial complex, so we'll get the same condition; around the vertex  $\odot_{\text{down}}$ , we see



yielding  $\frac{\lambda}{(1-\lambda)^2} \frac{\mu}{(1-\mu)^2} = 1$

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We shouldn't be surprised to see the same equation again: if we take the product around all the edges, each angle of each triangle is counted once; but the product around any triangle is equal to  $-1$ , and there are an even number of triangles. (Each tetrahedron contributes 4 triangles.)

[If we take logarithms and keep track of multiples of  $\pi i$ , an Euler characteristic argument shows that the equations are degenerate if the boundary is a torus and inconsistent otherwise.]

You may be surprised that the neighborhood of the vertex (its 'link') looks like a torus, rather than a sphere. Here are two ways to think about this:

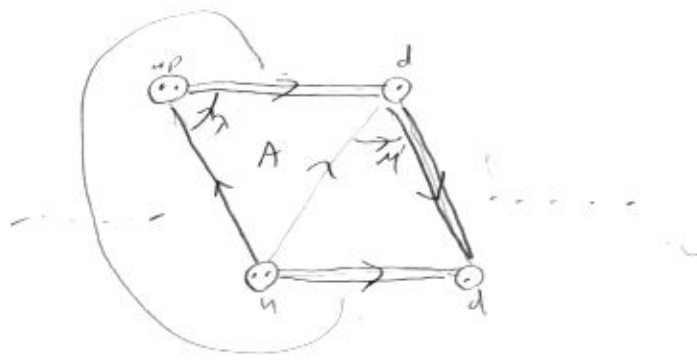
(a) One way to construct an ideal triangulation, at least combinatorially, is to take one of the triangulations Kashaev considers, with the link running through every vertex, and collapse all ~~edges containing the~~ tetrahedra containing at least one edge belonging to the link. For hyperbolic knots or links, you are guaranteed to be left with at least one tetrahedron. Then the link of the vertex will be the boundary of the complement of the knot, which is a torus.

(Note that some tetrahedra from this construction may end up negatively oriented. This is OK - you can still compute the volume solving the same equations.)

(b) Think of the complement as hyperbolic space modulo a discrete group. The cusp is a point ~~at~~ on the sphere at infinity, and a <sup>the bdy of a</sup> neighborhood looks like the bdy of a neighborhood in  $\mathbb{H}^3$  modulo the stabilizer of the point; i.e., a horosphere modulo a group of isometries. But the horosphere is Euclidean, so we ~~get~~ must get a torus. induced metric on the

From the second point of view, we see that the resulting torus should be flat in order to get a complete hyperbolic structure. In particular, the total rotation when you go around a non-trivial loop must be 0. Consider, for instance, the leftmost meridian in the figure I drew earlier:

~~(This is actually a meridian of the knot.)~~ (This is actually a meridian of the knot.)



We find that

$$\frac{1}{1-\mu} \cdot \frac{1}{1-\lambda^{-1}} = 1, \text{ or } \frac{\lambda}{(1-\mu)(1-\lambda)} = -1$$

Multiplying our previous equation (\*) by the square of this one, we find

$$\frac{\lambda}{\mu} = 1 \text{ or } \boxed{\lambda = \mu}$$

after which we get

$$(**) \quad \boxed{\lambda = -(1-\lambda)^2}$$

exercise Verify the solutions are  $\lambda = \mu = e^{\pm \pi i/3}$ , so our tetrahedra are equilateral, as ~~was~~ claimed on Friday.

~~How does this relate to the least action principle calculation?~~

We can calculate volumes using tetrahedra  $\Delta_\lambda$  with parameter  $\lambda$ , using the following result:

$$\text{Vol}(\Delta_\lambda) = \arg(1-\lambda) \log|\lambda| - \text{Im}(L_2(\lambda))$$

How does this relate to our earlier stationary phase calculations? Let's first rewrite the formula slightly (making a change of variables  $k \mapsto N-1-k$ ):

$$J_N(\mathbb{P}) = \sum_{k=0}^{N-1} \frac{(-1)^k q^{k(k+1)/2}}{[k!] [k!]}$$

Two things to note:

- There are two quantum factorials, each contributing a ~~stationary~~ dilogarithm with parameter  $q^k$  to the asymptotics; so we should have two tetrahedra with cross-ratio  $q^k$ ;
- the common cross-ratio satisfies the stationary phase condition

$$\frac{-1 \cdot q^k}{(1-q^k)^2} = 1$$

Note that this is the same equation as (\*\*).

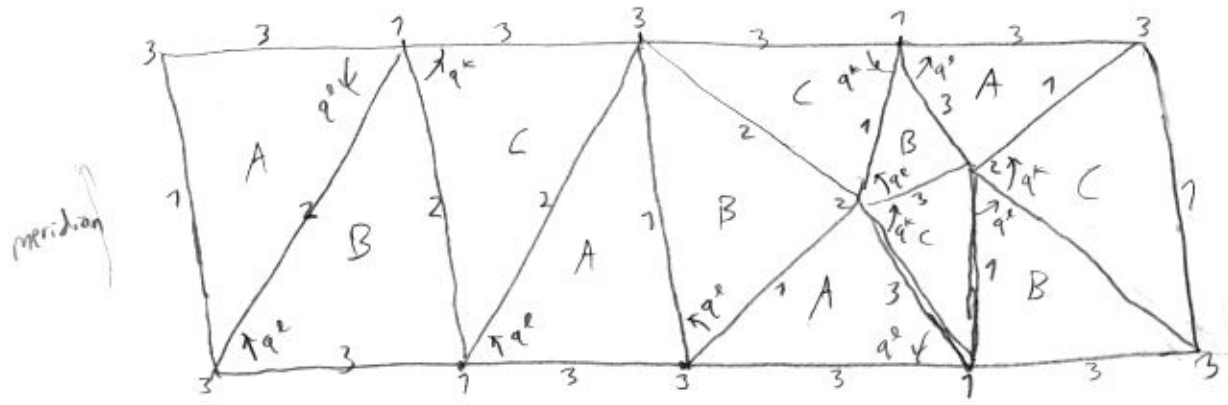
Is it a coincidence that we get precisely the same equations? in the two cases? I conjecture that it is not, we can at least see the same coincidence in other cases:

Exercise Compare Kashaev's formula for  $S_2$ ,

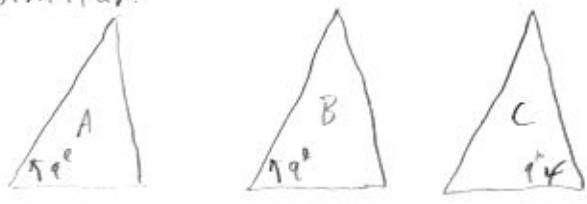
$$\langle \text{link} \rangle = \sum_{k \in \mathbb{Z}} \frac{[l!]^2}{[k!]^*} q^{-k(l+1)} = N^2 \sum_{0 \leq k+l \leq N} (-1)^k \frac{q^{l(l+1) + \frac{k(k+1)}{2} + lk}}{[l!]^2 [k!]}$$

with ~~the~~ an ideal triangulation of its complement.

Hint The neighborhood of the vertex in a triangulation of  $S^3 \setminus \text{link}$  looks like



(If I had drawn the figure correctly, all triangles would be similar:

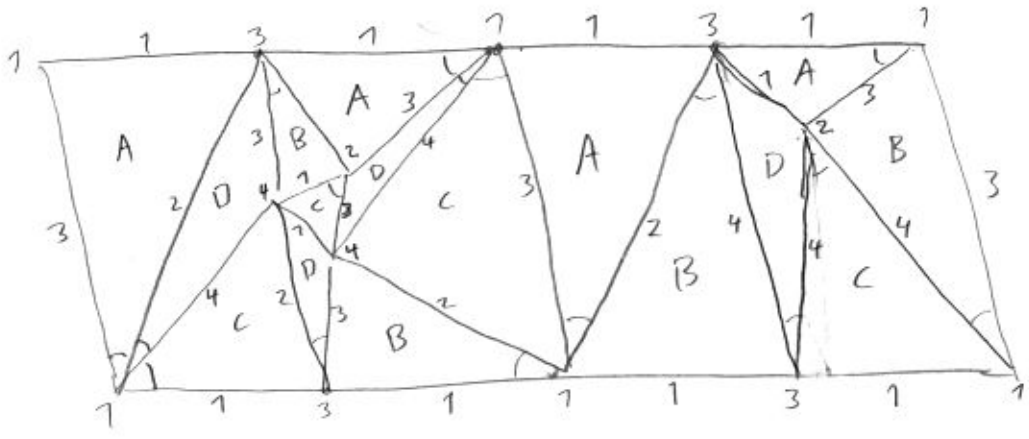


Note that the equations depend on the choice of parameters for the tetrahedra, so the angles I've indicated may be helpful (though there may be other choices giving the same equations.)

Extra Credit Do the same thing for the complement of the knot  $6_1$ :

$$\langle \text{link} \rangle = \sum_{k+l+m} \frac{[m!][m!]^*}{[k!][l!]^*} q^{(m+l)(m-k+1)} = N^2 \sum_{0 \leq k+l+m \leq N} (-1)^{m+l} \frac{q^{3\frac{m(l+1)}{2} + k(k+1) + \frac{l(l+1)}{2} + 2ml + ml + kl}}{[m!]^2 [k!] [l!]}$$

Hint The complement of  $b_1$  can be cut into 4 tetrahedra as follows:



(The marks are only there to indicate the orientation of ambiguous triangles, and are not suggestions of parameters.)

Exercise Play around with the program "Snap PPa" and figure out how to get pictures like those above and essentially anything else you want to know about hyperbolic manifolds.

Exercise Look in [Thur, Example 1.48, pp. 39-42] and see how to see that the two tetrahedra above do indeed give the knot complement. See [Thur, Examples 3.3.9-10, pp. 129-132] for the Whitehead link and the Borromean rings, and [BP, Section E.5-iv, pp. 210-222] for ~~more~~ a larger class (almost <sup>any</sup> alternating link).

Extra Credit See how one tetrahedron can make the complement of the trefoil, and compare the resulting <sup>matching</sup> equations with Le's ~~equations~~ <sup>formula</sup> for the Jones polynomial:

$$J'_\alpha(\mathcal{D}) = q^{1-\alpha} \sum_{k=0}^{\alpha-1} q^{-\alpha k} (1-q^{1-\alpha}) \dots (1-q^{k-\alpha})$$

[Neither set of equations will have solutions, but the sets of equations should be the same.]

## General Knots

In order to prove the conjecture, one will have to consider general knots, and not just particular examples. Here are some partial results in that direction.

One question is whether it is always possible to write down sums with the correct stationary points.

The answer seems to be yes. Consider the  $\mathbb{Z}$ -module generated by two variables per tetrahedron,  $\alpha_i$  and  $\beta_i$ , and define

$$f(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = \prod_i \frac{q^{\alpha_i \beta_i / 2}}{[\beta_i!]} \left[ = \prod_i \frac{1}{(q^{\alpha_i/2} - q^{-\alpha_i/2}) - (q^{\alpha_i/2} - q^{-\alpha_i/2 + \beta_i})} \right]$$

Each relation around an edge or a meridian gives us a relation, which we may write in the form

$$\prod_i \lambda_i^{x_i} (1 - \lambda_i)^{-y_i} = \pm 1$$

Assume all signs are + for simplicity, and consider the space

$$R = \text{span} \langle \sum x_i \alpha_i + \sum y_i \beta_i \mid x_i, y_i \text{ a relation} \rangle$$

and consider

(This is an  $n$ -dimensional space.)

$$\sum_{v \in R} f(v)$$

~~I'll show in a minute that the stationary phase points will be the solutions to the matching equations.~~

The formulas I wrote earlier are written for the projection of this space onto the space  $\langle \beta_i \rangle$ . Note that sums over the kernel of this map are easy.

This is definitely not quite right: I haven't specified the bounds of summation (which one doesn't see at all from stationary phase) and there are <sup>apparently</sup> some shifts by 1 in the correct answer. Also, the exponent of  $q$  is not always integer.

To see that stationary phase points give solutions to the matching equations for the hyperbolic structure, we need the following

Theorem [NZ, Theorem 2.2] If  $\langle \alpha_i, \beta_i \rangle$  is considered as a symplectic vector space with  $\alpha_i$  and  $\beta_i$  conjugate variables, then  $R$  is a Lagrangian subspace.

In particular, for any relation  $x_i, y_i$  and  $a_i \alpha_i + b_i \beta_i$  in the space  $\mathbb{R}$ , (15)

$$\sum (x_i b_i - y_i a_i) = 0.$$

At a critical point  $\sum a_i \alpha_i + b_i \beta_i \in \mathbb{R}$ , we furthermore have that the function  $f$  does not change when we add  $\sum x_i \alpha_i + y_i \beta_i$ :

$$\prod_i \frac{q^{(x_i b_i + y_i a_i)/2}}{(1 - q^{b_i})^{y_i}} = \prod_i \frac{q^{x_i b_i}}{(1 - q^{b_i})^{y_i}} = 1$$

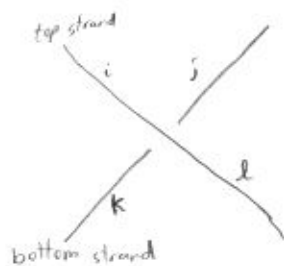
as desired.

[We can also see that we get the correct value at the stationary point, namely  $\exp\left(\frac{N}{2V_i} \left( \sum L_{i,j}(q^{b_i}) + \frac{1}{2} \log(q^{b_i}) \log(q^{a_i} = 1 - q^{b_i}) \right) \right)$ .]

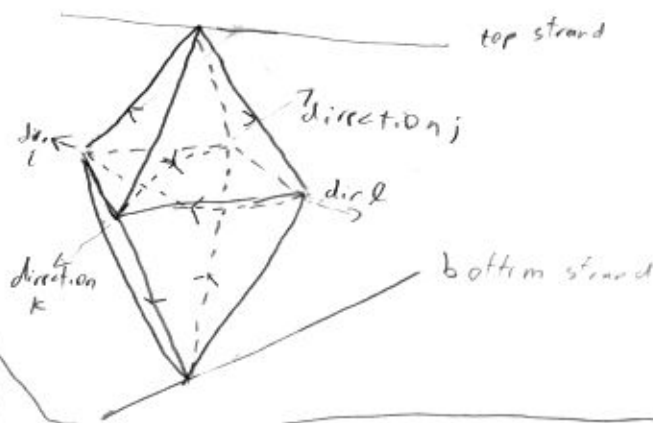
Another approach is to start with a standard  $R$ -matrix and see if we can reconstruct the corresponding tetrahedra. For instance, in [KM, Corollary 2.32], we find the following  $R$ -matrix (as quoted by [MN]):

$$R_{kl}^{ij} = \sum_{n=0}^{\min(N-1-i, j)} \delta_{l, i+n} \delta_{k, j-n} \frac{(i+n)! (N-1+n-j)!}{(n!) (i!) (N-1-j)!} q^{(i-\frac{n+1}{2})(j-\frac{n+1}{2}) - n(i-j) - n(n+1)/2}$$

We have 5 quantum factorials here, suggesting we should have 5 ideal tetrahedra in the answer. Furthermore, this invariant gives 0 unless we consider the 1-strand tangle version; this suggests we should use some auxiliary points on the strand where we broke it. We will do this, using two auxiliary points, one well above the plane of the crossing ( $+\infty$ ) and one well below ( $-\infty$ ).



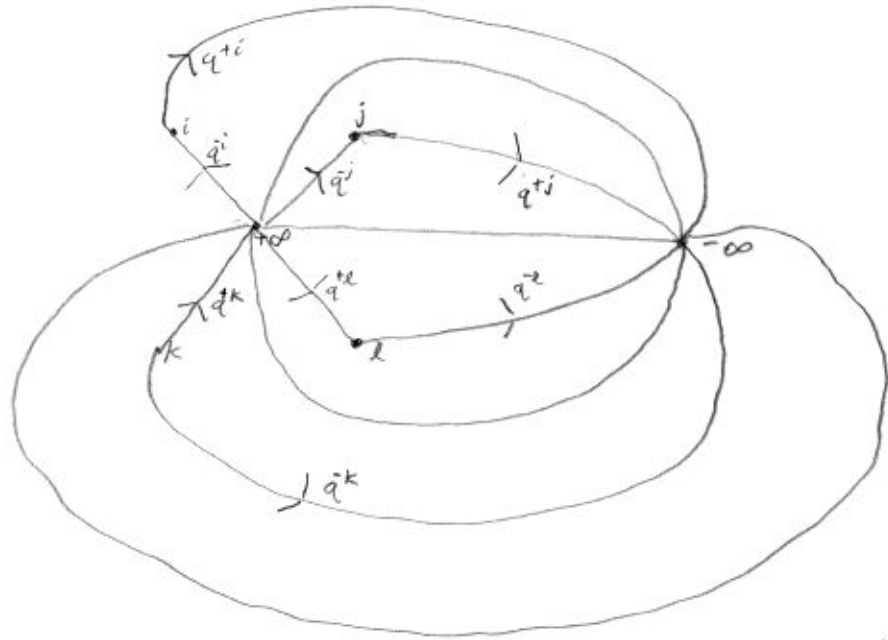
turn on side



On the right I have drawn a picture of this crossing turned on its side. Between the two strands, there is a new octahedron, which is divided into 5 smaller tetrahedra with the help of two edges, one from "direction  $k$ " to "direction  $j$ " above one from "direction  $i$ " to "direction  $l$ ". There is one tetrahedron in the middle (which will have cross-ratio  $q^n$ ) two on top (facing direction  $i$ , with cross-ratio  $q^i$ , and facing  $l$ , with cross-ratio  $q^l$ ), and two on bottom. Note that around the interior  $kj$  edge, for instance, we see that  $q^n \cdot q^i \cdot q^{-l} = 1$ , as desired.

(I'm cheating here, since I haven't specified orientations.)

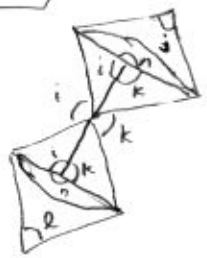
This is not yet a legal triangulation, since we have added 4 new vertices (which I will call  $v_i, v_j, v_k, v_e$ ) which are not on the knot. To fix this, drag vertices  $v_k$  and  $v_j$  around the upper strand and attach them to the auxiliary point  $+\infty$ , and drag vertices  $v_i$  and  $v_e$  down to  $-\infty$ . We get a strange gadget with a boundary that looks like this:



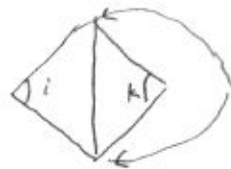
We see that we can glue these together identifying appropriate angles, as long as our strands don't go through critical points.

In general, these 3-dimensional pictures become unwieldy rapidly, and it's much easier to reason with cusp neighborhoods which I record here:

at  $+\infty$



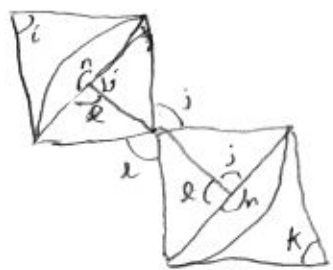
along upper strand



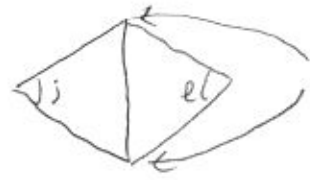
identify these two points

at  $-\infty$

[Caution: Orientation reversed]



along lower strand



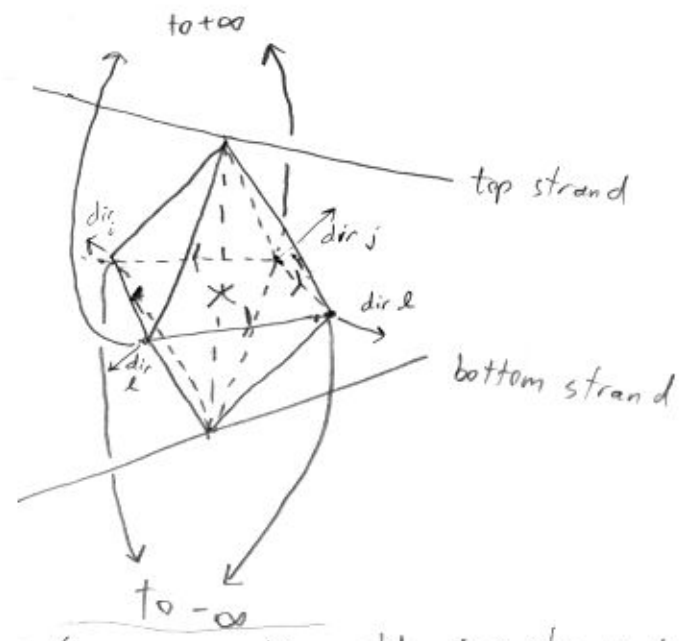
identify



Exercise Verify the assertions above. Add orientations to the last set of diagrams.

Extra Credit check that the quadratic factors give the correct equations for a hyperbolic structure at a critical point.

Exercise In [Kash 2], Kashaev gives an alternate R-matrix, involving 4 q factorials with parameters  $q^{i-j}$ ,  $q^{j-l}$ ,  $q^{l-k-1}$ , and  $q^{k-i}$ . Verify that this corresponds to the following decomposition:



i.e., add a single edge from the vertex on the top strand to the vertex on the bottom strand instead of the two we had earlier.

Verify the quadratic factors as well.

Extra Credit In [Kash 2], Kashaev claims that running the strand of a 1-strand tangle corresponds to setting the label on that strand to 0. Verify this claim. (Note: his context is slightly different.)

# The Colored Jones Polynomial

Let's formula for the figure 8,

$$J_q(\mathcal{B}) = \sum_{k=0}^{\infty} (1-q^{\alpha-k})(1-q^{\alpha-k+1}) \dots (1-q^{\alpha+k}) \cdot q^{-\alpha(k+\frac{1}{2})} = \sum_{k=0}^{\alpha-1} \frac{[\alpha+k!]}{[k+1!]} q^{-\alpha(k+\frac{1}{2})}$$

looks suggestively nice even when  $\alpha \neq N$ . (We have restored the factor  $(1-q^\alpha)$ , which is only relevant to the asymptotics if it happens to be 0.) Note that we have two quantum factorials, with parameters  $\alpha+k$  and  $\alpha-k-1$ ; however,  $[\alpha+k!]$  is in the numerator, which will contribute  $-Li_2(q^{\alpha+k})$  to the asymptotics, so we should really consider it to have parameter  ~~$N-\alpha-k-1$~~  using

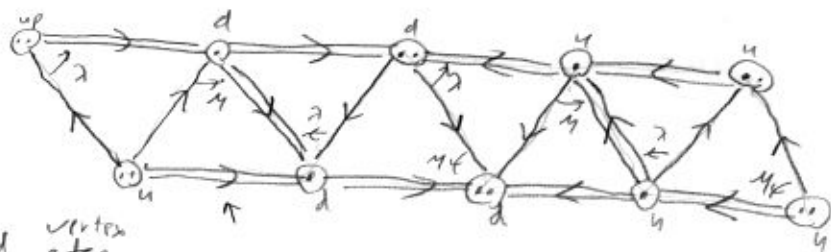
$$N \frac{(-1)^k q^{k(k+1)/2}}{[k!]} = [N-k-1!].$$

Furthermore, at the stationary point we will have

$$(1-q^{\alpha-k})(1-q^{\alpha+k}) = q^\alpha$$

Do these equations correspond somehow to hyperbolic geometry?

Recall the picture:



Around vertex  $\odot_d$ , we found the equation

$$\frac{(1-\lambda)^2}{\lambda} \frac{(1-\mu)^2}{\mu} = 1$$

and along the meridian, we had

$$\frac{\lambda}{(1-\mu)(1-\lambda)} = -1$$

Let's keep the first equation unchanged and introduce a parameter in the second:

$$\frac{\lambda}{(1-\mu)(1-\lambda)} = -\theta$$

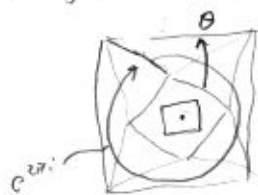
We then find

$$\frac{\lambda}{\mu} = \theta^2 \quad \frac{\lambda}{\theta} = \theta\mu =: \nu$$

$$-\theta = \frac{\lambda}{(1-\mu)(1-\lambda)} = \frac{\nu\theta}{(1-\frac{\nu}{\theta})(1-\nu\theta)} \Rightarrow (1-\frac{\theta}{\nu})(1-\nu\theta) = \theta$$

This agrees with the above with  $\nu = q^k$ ,  $\theta = q^\alpha$ .

This is a natural deformation of the equations in hyperbolic geometry. In particular, for appropriate values of  $\theta$ , the holonomy around another curve (e.g., the longitude) will become  $e^{2\pi i}$ ; we can then add a geodesic in the middle to get the surgered manifold. The Reshetikhin-Turaev sum over  $\alpha$  seems to achieve this naturally, but this is another story.



### Wild Speculation + Future Directions

Where next? It seems to me that there should be a simple formula for the colored Jones polynomial from any ideal triangulation of the complement. In proving this statement, the following theorem may be helpful:

Theorem (Pell): Any two combinatorial ideal triangulations of a cusped manifold  $M$  with at least two tetrahedra are related by a series of 3-2 moves:



~~For example, proving the conjecture is not as simple as it seems.~~

- It has been a long-standing problem to exactly solve CS theory with gauge group  $SL(2, \mathbb{R})$  or  $SL(2, \mathbb{C})$ , not least because it would be intimately related to quantum gravity, as in [Wit]. We see here suggestions that the answer has been staring us in the face. But it certainly needs to be more developed.
- If the conjecture is true, we can see a hyperbolic geometry from the Jones polynomial. As Boleau showed us last week, there are 7 other geometries of interest in 3 dimensions; all would probably have subexponential asymptotic growth. Can we see these geometries from features of the Jones polynomial / RT invariants?

The theory of Vassiliev invariants is, from one point of view, perturbation theory around the trivial connection (or stationary phase point). As we have seen in the past two weeks, it is a very rich theory. Here we have another stationary phase point which exists for all hyperbolic 3-manifolds (if the conjecture is true). What does perturbation around it look like? Does the perturbative series have good topological or geometric properties?

This is only a sample of the large number of questions opened up.

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