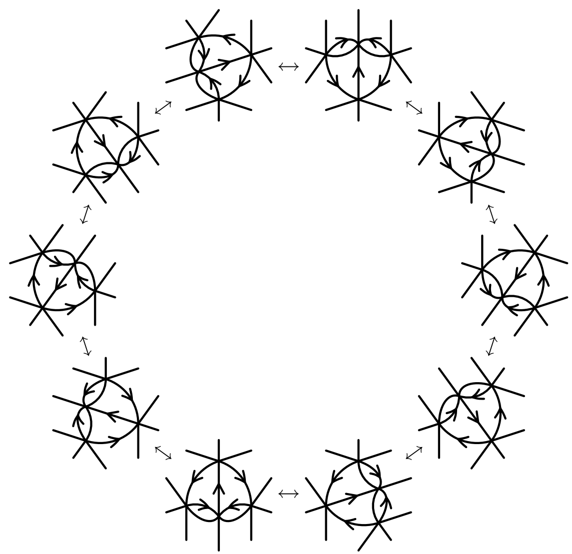
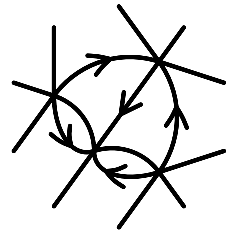
DOMINO TILINGS AND PLANAR ALGEBRAS GENERATED BY A 3-BOX

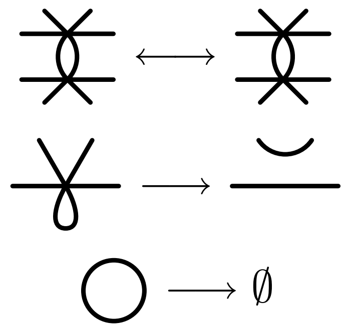
DYLAN THURSTON



A triple crossing diagram:

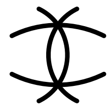
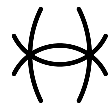
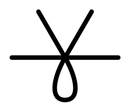
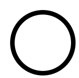


Basic moves between diagrams:



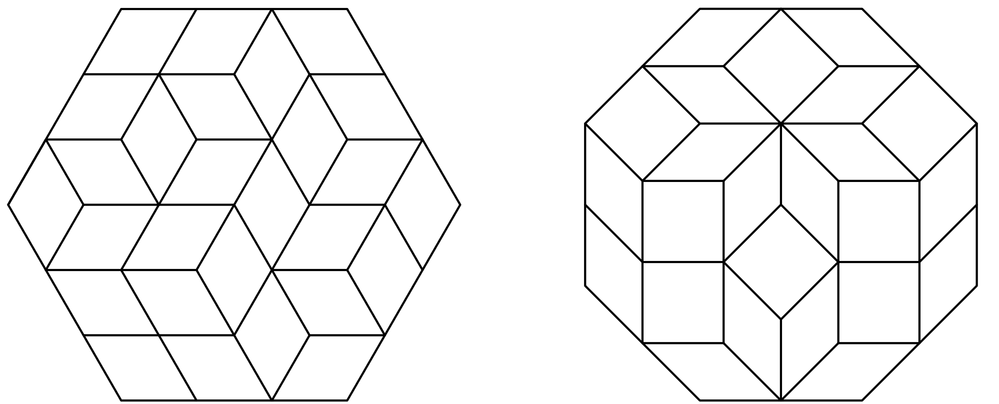
The main theorem:

Theorem. *For any two triple-crossing diagrams D, D', with the same connectivity when you follow the strands through the diagram, there is a third diagram D'' so that D and D' can both be reduced to D'' via a sequence of the basic moves:*

- $2 \leftrightarrow 2$ *moves*  $\leftrightarrow$ ;
- $1 \rightarrow 0$ *moves*  $\rightarrow$ ; *and*
- *Dropping a simple loop*  *with no crossings and an empty interior.*

Why do we care? **Tilings.**

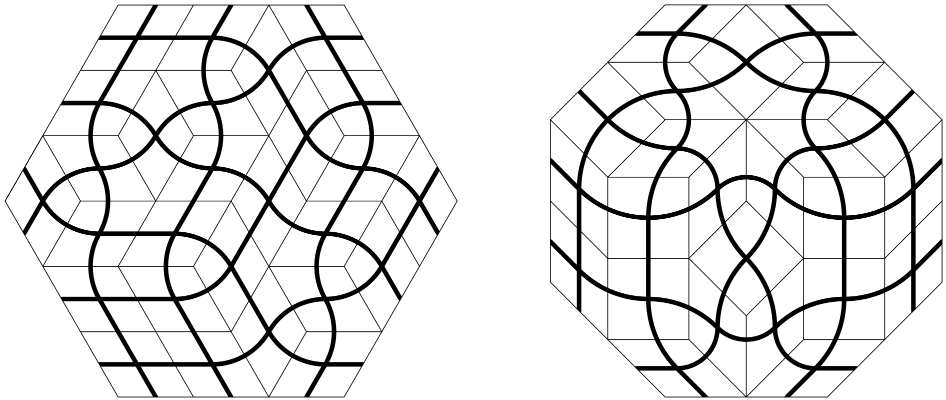
Lozenge tilings and their generalizations *rhombus tilings* have been extensively studied:



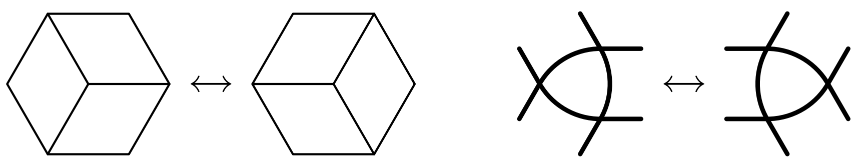
(A rhombus is a parallelogram with equal sides. A lozenge is a rhombus with angles 30-60-30-60.)

These tilings have deep connections in combinatorics and representation theory; for instance, the number of lozenge tilings of a hexagon are round numbers.

We can view these tilings topologically by running strands through the diagram connecting parallel edges.

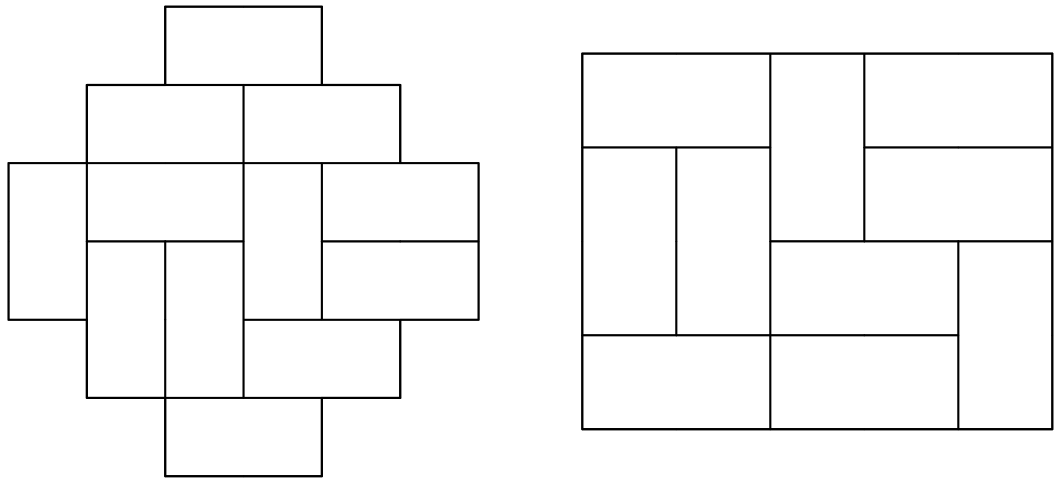


The different tilings of a given region are connected by a move like a Reidemeister move or the relation of the symmetric group. The connectivity of the strands is unchanged.



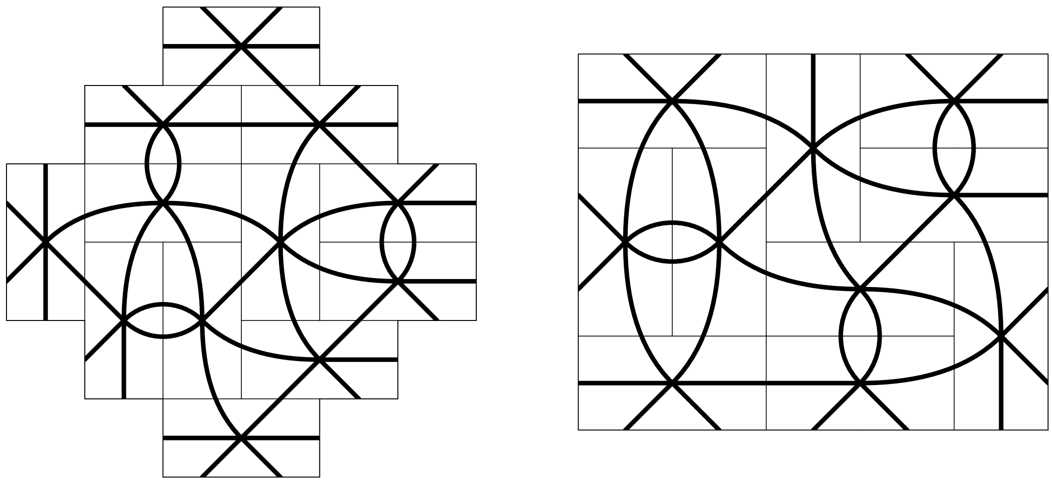
Note that strands that intersect parallel edges do not cross.

Domino tilings are another kind of tiling with rich connections in combinatorics and representation theory.

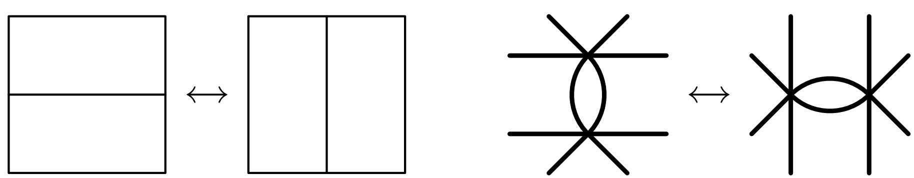


(A domino is a 1×2 rectangle.)

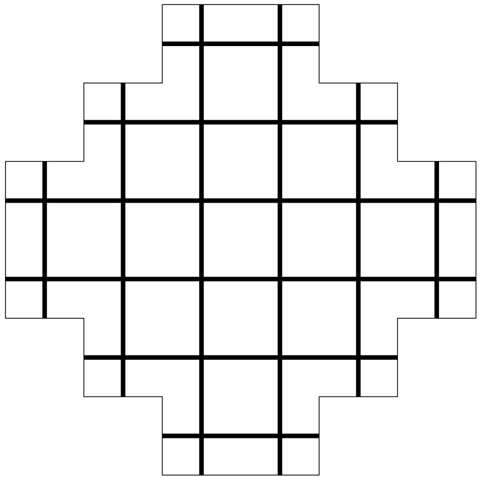
We can again make a more topological version with strands connecting opposite "sides" of each domino, where we think of a domino as having six sides of length 1.

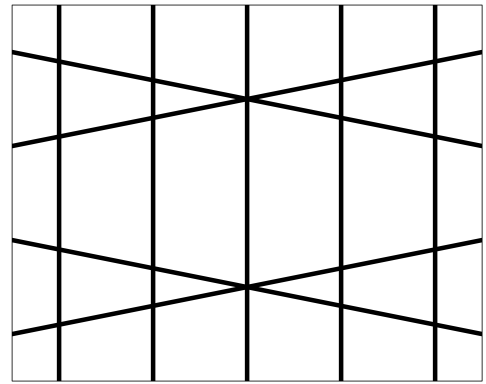


This time, the topological version of the basic move that connects tilings of a given region looks unfamiliar, but still leaves the connectivity of the strands unchanged.

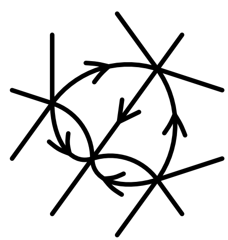


As a result, the total connectivity when we trace the strands across the diagram is an invariant of the region.

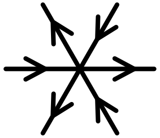




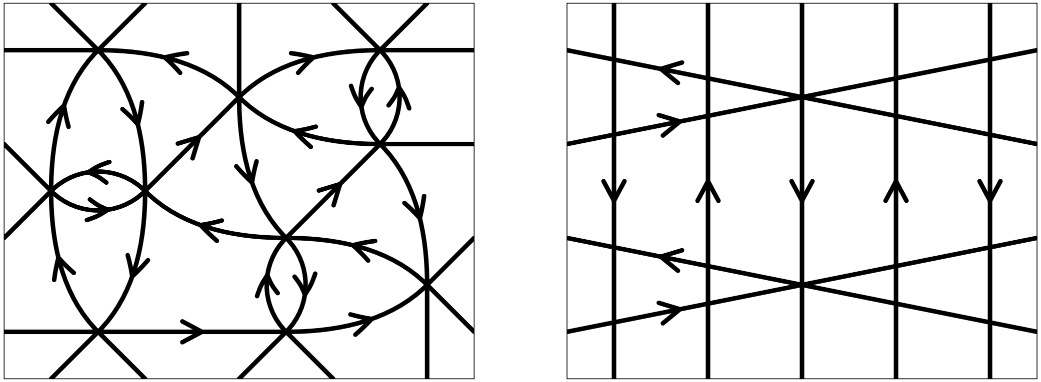
In the passage from lozenge tilings to rhombus tilings, we allowed more general ways to connect the strands. If we do the same thing here, we get a general *triple crossing diagram*: a collection of strands meeting three at a time.



Which connectivities are possible? There is at least one constraint. We can assign orientations to the strands alternating around a vertex.



We can continue this assignment to the whole diagram, with the orientations running alternately clockwise and counterclockwise around the complementary regions. Each strand has a consistent orientation, and the endpoints alternate in and out along the boundary.

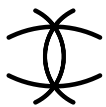
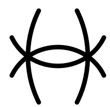
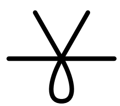
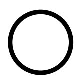


In general triple crossing diagrams, this turns out to be the only constraint.

Theorem. *In a disk with* $2n$ *endpoints on the boundary, all* $n!$ *pairings of in endpoints with out endpoints are achievable by some triple point diagram without closed strands.*

Furthermore, simple moves suffice to connect these diagrams.

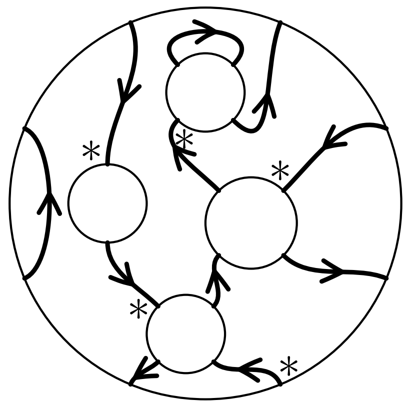
Theorem. *For any two triple-crossing diagrams* D, D', *with the same connectivity when you follow the strands through the diagram, there is a third diagram* D'' *so that* D *and* D' *can both be reduced to* D'' *via a sequence of the basic moves:*

- $2 \leftrightarrow 2$ *moves*  $\leftrightarrow$ ;
- $1 \rightarrow 0$ *moves*  $\rightarrow$ ; *and*
- *Dropping a simple loop*  *with no crossings and an empty interior.*

In particular, if D and D' both have the minimal number of triple crossings for the connectivity, they are connected by $2 \leftrightarrow 2$ moves.

Why do we care? **Planar algebras.**

A *planar algebra* consists of a complex vector space B_n for each n, the space of *n-boxes*. Each n-box conceptually has $2n$ strands attached to the boundary, alternating in and out. They may be joined together by oriented *arc diagrams* like this one:



This diagram, for instance, has 4 inputs (the little circles), with 4, 2, 4, and 4 strands, respectively; the output (the large circle) has 8 strands. This diagram therefore gives a multilinear map

$$B_2 \otimes B_1 \otimes B_2 \otimes B_2 \to B_4.$$

The $*$'s tell you how to rotate the boxes when you glue them. The elements we use in this paper will be symmetric, so they will not play a large role.

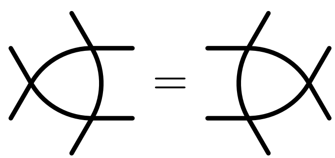
We can think of the passage from a geometric lozenge tiling to a rhombus tiling and on to general arc diagrams as a way of turning lozenge tilings into planar algebras: we drop the geometric restriction on how we may place two lozenge tilings together. That is, the shape of the boundary no longer needs to match.

Similarly, triple crossing diagrams are a way of relaxing domino tilings into a planar algebra.

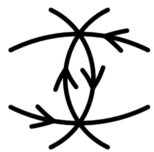
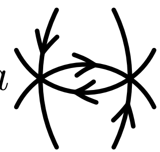
Every planar algebra contains as a subalgebra the arc diagrams with no input; that is, collections of non-crossing strands in the disk. This algebra is called the *Temperley-Lieb algebra*. To go beyond the Temperley-Lieb algebra, we look for algebras which can be expressed in terms of a finite number of n-boxes, for different values of n.

Bisch and Jones studied planar algebras generated by a single 2-box. An element generated by this single 2-box looks like the crossing strands dual to a rhombus tiling: Insert the generating 2-box at each crossing.

Bisch and Jones classified all planar algebras generated by a single 2-box and where the dimension of the space of 3-boxes is less than or equal to 14. This assumption implies the existence of relations like this one:


+ lower order terms

We now turn to diagrams generated by a single 3-box. We can represent them by a triple-crossing diagram of the type we have been considering. There are 26 different diagrams in a 4-box that can be constructed with no more than 2 3-boxes. If we assume that the dimension of the space of 4-boxes is less than or equal to 24, then generically we may expect a relation of the form

 $= a$  $+$ lower order terms

and the orientation reverse.

Corollary. *If relations of the form above hold in a planar algebra generated by a single 3-box, then*

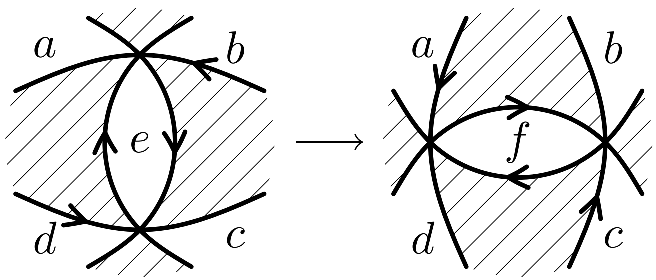
$$\dim B_n \leq n!$$

There are interesting examples where the bound is tight: planar algebras related to the HOMFLY polynomial have exactly this dimension.

Why do we care? **Cluster algebras.**

There is a cluster algebra naturally associated to every minimal triple-crossing diagram. We will state the results directly.

Color the regions of a triple crossing diagram black and white, and assign a variable to each white region. When we perform a $2 \leftrightarrow 2$ move with a central white region, assign a variable to the new region as follows.



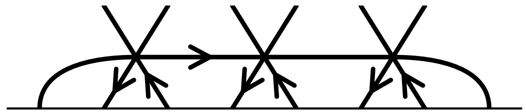
$$f = \frac{ac + bd}{e}.$$

Then, starting from a minimal diagram D and going to a minimal diagram D', the rational functions on the regions of D' are:

- Independent of the path from D to D' and
- Laurent polynomials in the variables of D with positive coefficients.

The proof: **Generating diagrams**

Given an arbitrary pairing of $2n$ in and out endpoints. Pick one pair, and run a strand parallel to the boundary, crossing over strands on the way in pairs:



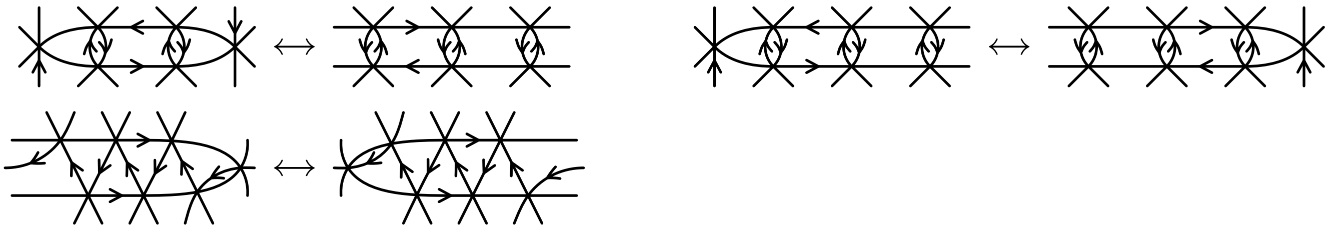
Note that there will always be an even number of strands to cross over because of the alternation of in and out endpoints.

If we can fill in the remainder of the diagram with a suitable pairing of the remaining $2n-2$ endpoints, we will construct the desired original pairing. By induction, this is possible.

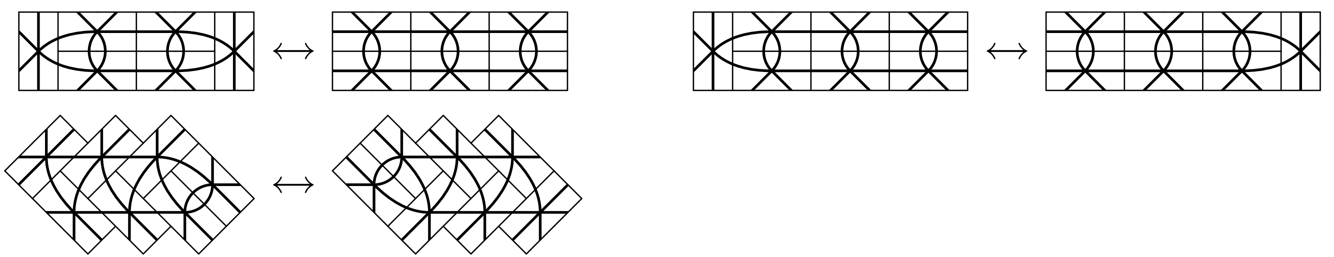
This procedure will sometimes introduce unnecessary extra crossings. If we ensure that we do not run the strand over a nested strand, we will see that the diagram we construct is minimal (i.e., has the minimal number of triple crossings for the given permutation.) A diagram constructed this way will be called *standard*.

The proof: **Useful moves**

Lemma. *Each of the following triple-point diagrams can be related by sequences of* $2 \leftrightarrow 2$ *moves:*



Proof. In each case, the two triple-point diagrams come from two different domino tilings of the same region:

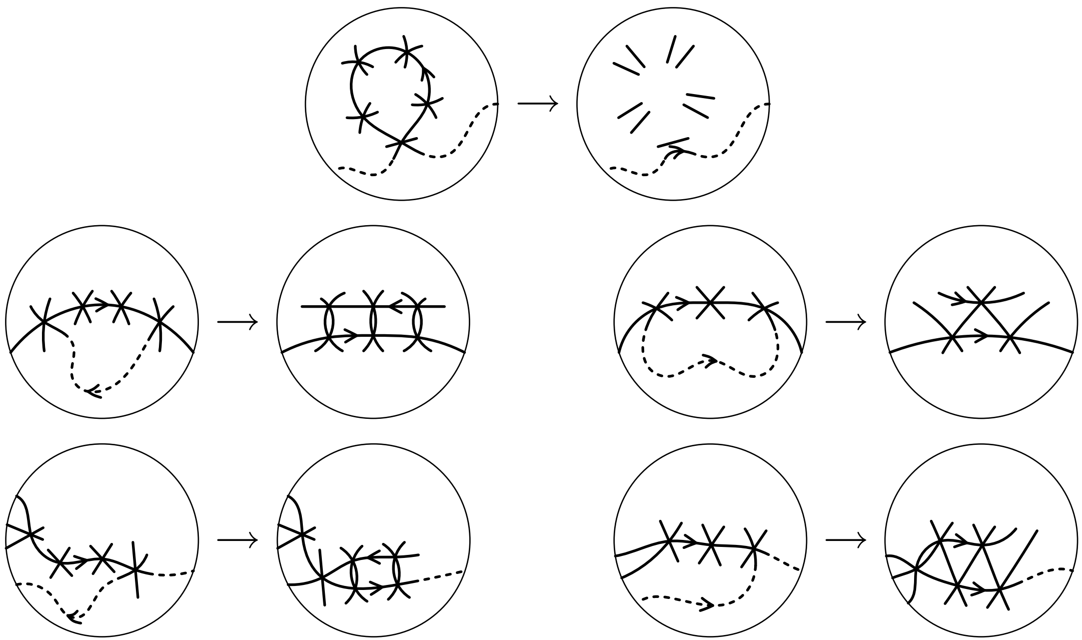


Any two domino tilings of the same region can be related by a sequence of domino flips. □

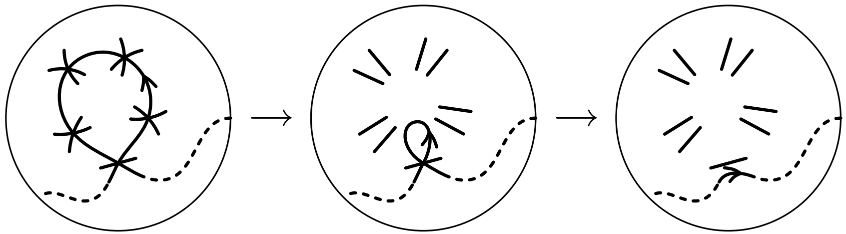
The proof: **Key lemma**

Lemma. *Let D be a triple diagram and let S be a strand of the associated pairing which does not enclose another pairing. Then D is related, by a sequence of* $2 \leftrightarrow 2$, $1 \to 0$ *and loop dropping moves, to a diagram D' in which S is boundary-parallel.*

The proof proceeds by successively straightening S:



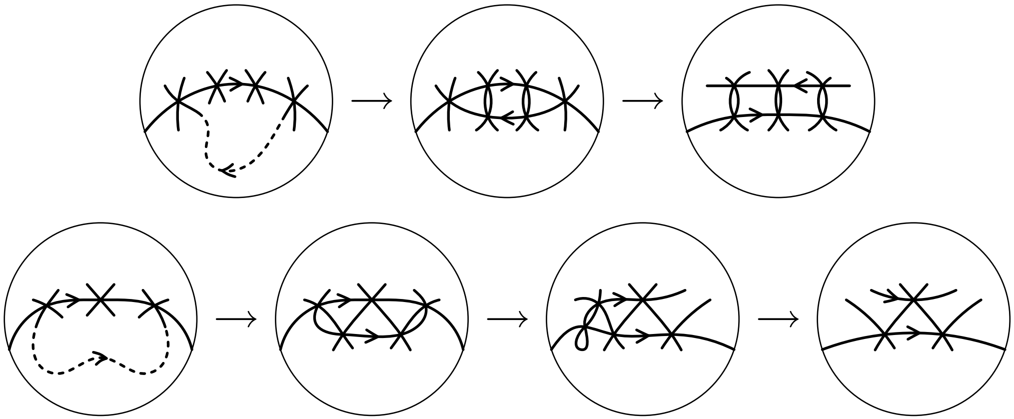
Step 1. Remove self-intersections of S.



Take an innermost loop L of S. The region inside L (including the boundary, except for the self-intersection), has fewer triple crossings, so by induction we can make L boundary parallel. A single further $1 \to 0$ move removes the original self-intersection of S.

Repeat this step until no self-intersections of S remain.

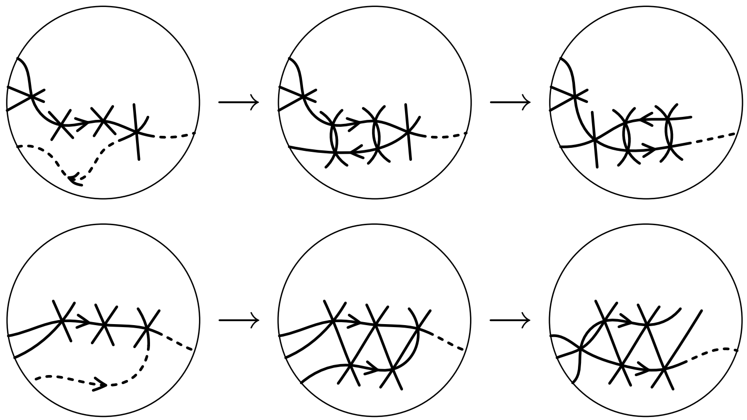
Step 2. Remove double intersections with S.



If there is some strand that intersects S at least twice, then consider a minimal one T. Because there are fewer triple crossings strictly inside of S than there were in D, we can by induction apply moves until T is boundary-parallel. Depending whether T and S are parallel or anti-parallel, we can apply a sequence of $2 \leftrightarrow 2$ moves from the lemma. If they are parallel, we additionally apply two $0 \to 2$ moves. In either case, we end up with no intersections between T and S.

Repeat this step until there are no strands that intersect S twice. The number of triple intersections along S strictly decreases at each step, so this terminates.

Step 3. Comb out triple crossings.



At this point, the only strands left between S and and the boundary come from the boundary and cross S. (Remember no strands connect the boundary within S to itself.) To make S boundary parallel, want all the strands to run directly to S.

Consider the strand T with the leftmost endpoint which is not straight. condition. By induction, make T boundary parallel to S. Depending whether T is parallel or anti-parallel to S, apply one of the moves from the lemma to make T run straight to S.

Repeat this step until every strand runs directly from I to S.

Step 4. Remove disconnected components. Now S is boundary parallel, with the exception of possible disconnected components of the diagram between S and the boundary. For each disconnected component, consider a region R that encloses all of the component except for one arc T on the boundary of the component. The diagram contained inside R has no more triple crossings than D and at least one fewer loop, so by induction we can reduce it until the unique arc runs boundary parallel on the side facing T. This strand forms a simple loop with T, which we can drop.

Repeat this step until there are no disconnected components between S and the boundary, and so S is boundary parallel. The total number of loops in the diagram decreases at each step, so this terminates. □

The proof: **Proving the theorem**

Given D and D' with the same connectivity, pick a standard representative D'' also with the same connectivity. Consider an outermost strand of D''. By the Key Lemma, we can reduce D until that strand agrees with D''. Repeat with the smaller diagram until we have reduced D to D''. We can do the same for D', proving the theorem.

Further notes: **Relations between relations**

Conjecture. *The following diagrams give all relations between relations.*