Computing with curves on surfaces

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Goal

Understand the *intersection numbers* and *Dehn-Thurston coordinates* of curves on surfaces.

- Easy algorithms for mapping class group (the “arithmetic field” of the surface)
- Pick random curves
- Compute stable/unstable measured laminations
- Relation to hyperbolic geometry
- Convexity

Setting: Intersection numbers

A *multi-curve* is an immersed 1-manifold on a surface Σ (transverse intersections).

A multi-curve is *simple* if it has no self-intersections.

A multi-curve is *taut* if the number of self-intersections is minimal in the homotopy class. Equivalent: no (homotopic) bigons or monogons.

Convention: lower-case letters for concrete curves, capital letters for homotopy classes.

Intersection numbers:

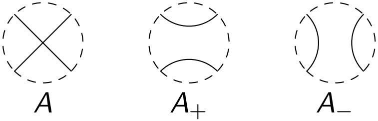
$$\begin{aligned} i(a,b) &= \#(a \cap b) \\ i(A,B) &= \text{minimum in homotopy class} \end{aligned}$$

The key lemma

Smoothing Lemma (Topological version)

Let x be a simple multi-curve, a be a taut multi-curve, a_+, a_- be smoothings of a. Then

$$i(X, A) = \max(i(X, A_+), i(X, A_-)).$$



- All curves are multi-curves unless otherwise specified.
- We are interested in parameterizing X by varying A.
  Will write $i(X, A) = i_X(A) = iA$.

Variations

- On a surface Σ with punctures, can allow the curve a to run to *punctures*. (Then a is a proper, immersed arc.)

Smoothing Lemma (Topological arc version)

Let x be a simple multi-curve, a be a taut multi-curve/arc, a_+, a_- be smoothings of A. Then

$$i_X(A) = \max(i_X(A_+), i_X(A_-)).$$

- Can also allow x to contain arcs with distinct endpoints. Convention: x may run to the *boundary*. (Then x is an immersed compact 1-manifold with boundary.)
- Can resolve multiple crossings at once, take the maximum over all smoothings.

Parametrizing curves

Let T be a triangulation of a surface with punctures, vertices at punctures.

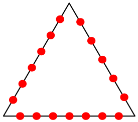
Theorem

Simple curves X are determined by $(i_X(E))_{E \in T}$.

Proof. Any triple of intersections with sides of a triangle that:

- satisfies the triangle inequalities; and
- has even total parity

has a unique filling.



There is a unique way to match up adjacent triangles.

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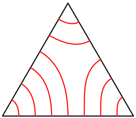


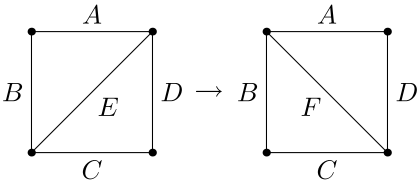
There is a unique way to match up adjacent triangles.

Quadrilateral flip

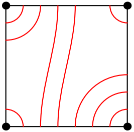
How are numbers $(iE)_{E \in T}$ related for different triangulations?

Lemma

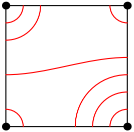
In a quadrilateral as below, $iE + iF = \max(iA + iC, iB + iD)$.



Proof 1. By cases on X:



$iA + iC$ bigger



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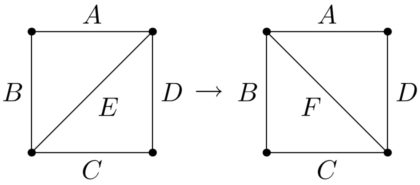
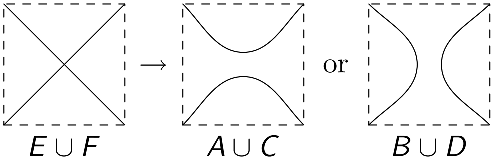


Proof 2. Consider $E \cup F$ as a single curve and apply the Smoothing Lemma.



Proof sketch

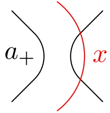
Smoothing Lemma (Topological version)

Let x be a simple multi-curve, a be a taut multi-curve, a_+, a_- be smoothings of A. Then

$$i(X, A) = \max(i(X, A_+), i(X, A_-)).$$

Proof. Assume $a \cup x$ is taut and all intersections are generic.
Then $i(X, A) \geq i(X, A_+)$ and $i(X, A) \geq i(X, A_-)$.
To show: one inequality is an equality.

If $i(X, A_+) < i(X, A) = i(x, a_+)$, then
$x \cup a_+$ must have an (x, a_+) bigon. This bigon must include the point of smoothing. Since x is simple, we can pass to an innermost bigon (measured along a_+).



If $x \cup a_-$ is also not taut, the two bigons combine to give a (x, a) bigon, contradicting tautness.

Proof sketch

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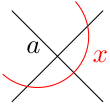

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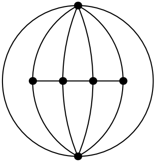
The word problem in the braid group (Dynnikov)

Take a triangulation of the disk with k punctures.



Apply an element g of the braid group and perform flips to get back to the original triangulation.

Follow a curve C along to compute gC. If $gC \neq C$, then $g \neq \mathrm{id}$.

If $gC = C$ for sufficiently many C, then $g = \mathrm{id}$.

This solves the word problem in the braid group in time quadratic in the length of the word, independent of the number of generators.

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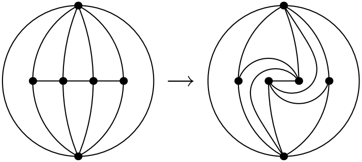

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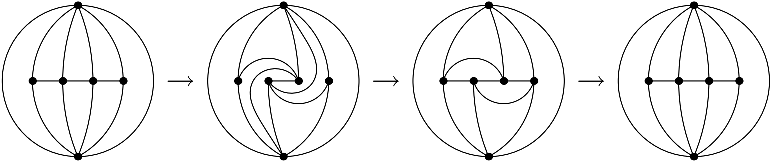

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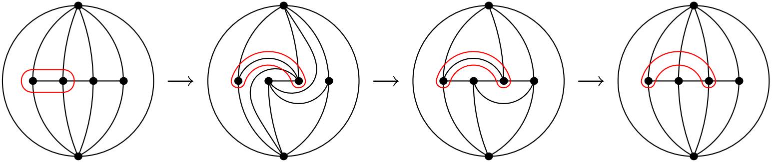

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Convexity

For a triangulation T, $i_X(E)$ for $E \in T$ determines X. So for every A, $i_X(A)$ is a function of the $i_X(E)$. What function?

Proposition

iA is a convex, piecewise-linear function of $(iE)_{T \in E}$.

Proof. If A is not simple, $iA = \max(iA_+, iA_-)$.

If A is simple and not an edge of T, pick an edge E so that $A \cap E \neq \emptyset$. Then

$$iA = i(A \cup E) - iE = \max(iB) - iE$$

where maximum is over all smoothings B of $A \cup E$.
Each smoothing has fewer intersections with E,
no more intersections with other edges of T.

By induction reduce to simpler case. $\square$

Hyperbolic geometry: Penner's λ lengths

For Σ a hyperbolic surface (with a horocycle marked around each puncture), a an immersed curve on Σ, define

$$\lambda_\Sigma(a) = \lambda a = \begin{cases} \lambda a_1 \cdot \lambda a_2 & a = a_1 \cup a_2 \\ \pm 2\cosh(la/2) & a \text{ an immersed } S^1 \\ \pm \exp(la/2) & a \text{ an immersed arc} \\ -2 & a \text{ trivial circle} \end{cases}$$

Here la is the length of a, measured between horocycles in case a is an arc. λa is positive if a has no monogons (trivial loops).

If a is a circle, $\rho(a)$ corresponding element of SL_2, then

$$|\lambda a| = |\operatorname{tr}(\rho(a)|.$$

Smoothing Lemma (Geometric version)

If a is any curve, a_+ and a_- smoothings of a, then

$$\lambda a = \lambda a_+ + \lambda a_-.$$

Geometric proof sketch

Smoothing Lemma (Geometric version)

If a is any curve, a_+ and a_- smoothings of a, then

$$\lambda a = \lambda a_+ + \lambda a_-.$$

Proof. In SL_2, a matrix satisfies its characteristic equation:

$$\begin{aligned} A - \operatorname{tr}(A) + A^{-1} &= 0 \\ AB - \operatorname{tr}(A)B + A^{-1}B &= 0 \\ \operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) &= \operatorname{tr}(A)\operatorname{tr}(B). \end{aligned}$$

The relation between $\operatorname{tr}(\rho(a))$ and λa depends on a lifting from PSL_2 to SL_2, a spin structure; the signs work out as stated.

Algebraic interlude: Cluster algebras

Define a sequence by

$$x_0, x_1 \text{ arbitrary positive reals}$$
$$x_{n+1}x_{n-1} = x_n + 1$$

Observation.

$$x_{n+5} = x_n$$

Proof.

$$x_2 = \frac{x_1}{x_0} + \frac{1}{x_0} \qquad x_3 = \frac{1}{x_0} + \frac{1}{x_1 x_0} + \frac{1}{x_0} \qquad x_4 = \frac{x_0}{x_1} + \frac{1}{x_1} \qquad x_5 = x_0$$

But why?

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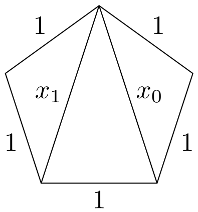
Moduli of ideal hyperbolic pentagons: Can choose horocycles around vertices so they just touch; moduli given by lengths of two diagonals.



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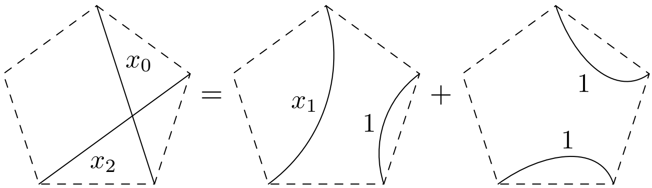


$$x_0x_2 \quad = \quad x_1 \quad + \quad 1$$

Relation: Geometry to topology

Measured laminations are the space at infinity of Teichmuller space, so intersection numbers transform as the *tropical limit* of lengths of curves.

Given a simple curve X on Σ, we can insert a long cylinder at X, to get a surface Σ'.

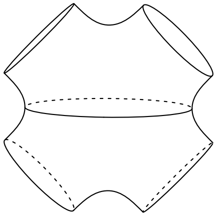
If the neck is long, the hyperbolic length of another curve A will be dominated by number of intersections with X.

$$\begin{aligned}
l_{\Sigma'}(A) &\sim C \cdot i(X, A) \\
\lambda_{\Sigma'}(A) &\sim k^{i(X,A)} \\
\text{multiplication} &\rightsquigarrow \text{addition of exponents} \\
\text{addition} &\rightsquigarrow \text{max of exponents} \\
\lambda A = \lambda A_+ + \lambda A_- &\rightsquigarrow iA = \max(iA_+, iA_-)
\end{aligned}$$

Closed surfaces

Closed surfaces are generally more difficult than surfaces with at least one puncture. In particular, the mapping class group is harder to deal with, and it is harder to parametrize curves.

In *Dehn-Thurston coordinates* for curves, the surface is cut up as much as it can be along simple curves. Now the elementary pieces are pairs of pants rather than triangles.



Again the *intersection numbers* determine the curve uniquely inside a pair of pants, as long as the total parity is even.

The ways to glue the curves in two adjacent pants are controlled by another integer, the *twist parameter*. This depends on some further choices and marking.

How do parameters change when pair of pants decomposition is changed?

Pants transformation, new version (T)

```
lk a b c = 0 `up` ((a+b-c)//2) `dn` a `dn` b
secondTrans (ma,mb,mc,md) e =
  let me = m e ; te = t e
      f x y z = 0 `up` (y - x) `up` (z - x) `up`
                  ((y+z-x)//2)
      m1 = (ma + mc - me) `up` (mb + md - me)
      m2 = (abs te + f me ma mb + f me mc md)
      mf = m1 `up` m2
      tf = if m1 > m2 then -te
           else -sign te*(me - f mf mb mc - f mf md ma)
      la = lk ma me mb ; lb = lk mb me ma
      lc = lk mc me md ; ld = lk md me mc
      dta = sign te * lk (abs te) la ld
      dtb = sign te * lk (abs te) lb lc
      dtc = sign te * lk (abs te) lc lb
      dtd = sign te * lk (abs te) ld la
  in (DTAnnulus { m = mf, t = tf }, (dta, dtb, dtc, dtd))
```

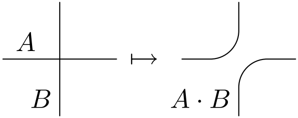
Pants transformation, old version (Penner 1984)

```
secondTrans (m3,m2,m4,m5) a1 =
  let [a2,a3,a4,a5] = map dummyAnnulus [m2,m3,m4,m5]
      BasicPants { l11 = l11, l22 = l22, l33 = l33,
                   l12 = l12, l23 = l23, l13 = l13 }
         = dtToBasicPants a1 a3 a2
      BasicPants { l11 = k11, l22 = k22, l33 = k33,
                   l12 = k12, l23 = k23, l13 = k13 }
         = dtToBasicPants a1 a4 a5
      t1 = t a1
      k11' = k22 + l33 + (vL - k13) `up` 0 + (-vL - l12) `up` 0
      k22' = (vL `dn` l11 `dn` (k13 - l12 - vL)) `up` 0
      k33' = ((-vL) `dn` k11 `dn` (l12 - k13 + vL)) `up` 0
      k23' = (k13 `dn` l12 `dn` (k13 - vL) `dn` (l12 + vL)) `up` 0
      k12' = -2*k22' - k23' + k13 + k23 + 2*k33
      k13' = -2*k33' - k23' + l12 + l23 + 2*l22
      l11' = l22 + k33 + (vK - l13) `up` 0 + (-vK - k12) `up` 0
      l22' = (vK `dn` k11 `dn` (l13 - k12 - vK)) `up` 0
      l33' = ((-vK) `dn` l11 `dn` (k12 - l13 + vK)) `up` 0
      l23' = (l13 `dn` k12 `dn` (l13 - vK) `dn` (k12 + vK)) `up` 0
      l12' = -2*l22' - l23' + l13 + l23 + 2*l33
      l13' = -2*l33' - l23' + k12 + k23 + 2*k22
      dt2 = l33 + ((l13 - l23' - 2*l22')`dn`(vK + l33' - l22')) `up` 0
      dt3 = -k33' + ((vL + k33' - k22')`up`(k23' + 2*k33' - l12)) `dn` 0
      dt4 = -l33' + ((vK + l33' - l22')`up`(l23' + 2*l33' - k12)) `dn` 0
      dt5 = k33 + ((k13 - k23' - 2*k22')`dn`(vL + k33' - k22')) `up` 0
      s = sgn (vL + vK + l33' - l22' + k33' - k22')
          (l12 - 2*k33' - k23' /= 0)
      t1' = k22 + l22 + k33 + l33 - (l11' + k11' + dt2 + dt5) +
        s*(t1 + l33' + k33')
      vL = l11 + t1
      vK = k11 + t1
  in (DTAnnulus { m = 2*l11' + l12' + l13', t = t1' } , (dt3, dt2, dt4, dt5))
```

Multiplication on curves (Luo 1999)

To understand the twist parameters, we first need to understand a type of multiplication of curves.

There is one special smoothing of $A \cup B$, called the *product* $A \cdot B$:



This can be thought of as shifting B one spot to the right along A. In particular, if we do it enough times we get a Dehn twist:

$A^{i(A,B)} \cdot B = A \cdot (\cdots (A \cdot (A \cdot B)) \cdots) =$ Dehn twist of B around A

Definition. $A^n \cdot B = B \cdot A^{-n}$ if $n < 0$.

Theorem (Luo)

If A, B, X are simple and X has no arc components, $i(X, A^n \cdot B)$ is a convex function of n.

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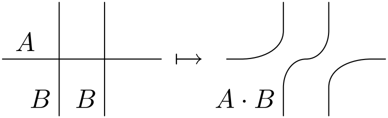

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The twisting number

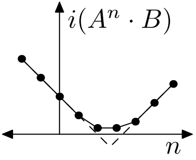
Lemma

The absolute value of the slope of $i(X, A^n \cdot B)$ is $\leq i(X, A)$.

Proof.

$$\begin{aligned} i_X(A^n \cdot B) &= i_X(A^n \cdot B \cup A) - i_X(A) \\ &= \max(i_X(A^{n+1} \cdot B), i_X(A^{n-1} \cdot B), \ldots) - i_X(A) \end{aligned}$$

Bounds are achieved for large or small n.
The *twisting* is the offset of the minimum.



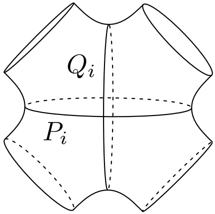
Definition

$$\mathrm{tw}_X(A; B) = \lim_{n \to \infty} i_X(A^n \cdot B) - i_X(B \cdot A^n).$$

Dehn-Thurston coordinates

Pick a pants decomposition P, with a maximal collection of simple loops P_i, $i = 1, \ldots, k$. For each i, pick a dual curve Q_i, so that

$$i(P_i, Q_j) = 2\delta_{ij}$$



Definition

The *Dehn-Thurston coordinates* for a simple curve X are

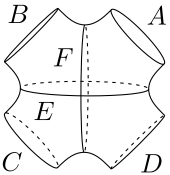
$$(m_i, t_i)_{i=1,\ldots,k} = (i_X(P_i), \mathrm{tw}_X(P_i; Q_i)).$$

These differ from previous definitions by a shift and scale in the twist parameter.

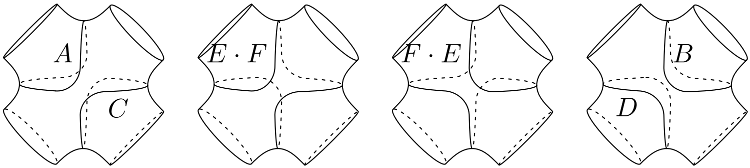
Change of coordinates

Question. Given $iA, \ldots, iE$ and tE, what is iF?

As before
consider $E \cup F$ and apply the smoothing lemma:

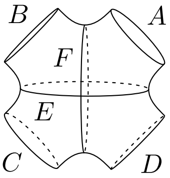


$$\begin{aligned} iE + iF &= \max(iA + iC, i(E \cdot F), i(F \cdot E), iB + iD) \\ iF &= \max(iA + iC - iE, iB + iD - iE, |tE| + \text{height of vertex}) \\ &= \max(\, iA + iC - iE, iB + iD - iE, \\ &\qquad |tE| + \ \max(0, (iA + iB - iE)/2, iA - iE, iB - iE) \\ &\qquad\qquad \max(0, (iC + iD - iE)/2, iC - iE, iD - iE)). \end{aligned}$$



Change of coordinates

Question. Given $iA, \ldots, iE$ and tE, what is iF?



As before
consider $E \cup F$ and apply the smoothing lemma:

$$
\begin{aligned}
iE + iF &= \max(iA + iC, i(E \cdot F), i(F \cdot E), iB + iD) \\
iF &= \max(iA + iC - iE, iB + iD - iE, |tE| + \text{height of vertex}) \\
&= \max(\, iA + iC - iE, iB + iD - iE, \\
&\qquad |tE| + \ \max(0, (iA + iB - iE)/2, iA - iE, iB - iE) \\
&\qquad\qquad \max(0, (iC + iD - iE)/2, iC - iE, iD - iE)).
\end{aligned}
$$

The height of the extrapolated vertex turns out to be the number of intersections with X of a curve from E to E in the top half plus the corresponding number for the bottom half, which is a maximum of 4 terms by a case analysis.

Further directions

- Find general principles to avoid the last case analysis.
- Find a geometric version of the twist principle used. (See [Okai 1993].)
- Can you solve the conjugacy problem in the MCG? You can easily find the stable and unstable laminations; but how to tell if two pairs are topologically conjugate?
- Normal surfaces: the change of coordinates for the quadrilateral flip is closely connected to normal surfaces. Is there a corresponding theory coming from the change of coordinates in the pair of pants?
- There are corresponding theories for representations into SL_n, $n > 2$. What is the geometric meaning of the tropical limit?