Combinatorial link Floer homology

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Outline

4-dimensional invariants

Several theories giving 4-manifold invariants:

Donaldson theory
$\leftrightarrow$ (conj. Seiberg–Witten '94)
Monopoles (Seiberg-Witten)
$\cong$ (Taubes '08)
Embedded contact homology (ECH)
$\cong$ (Cutluhan-Lee-Taubes, Colin-Ghiggini-Honda '10)
Heegaard Floer (HF) homology

We topologists only have one trick in 4 dimensions!

Each theory has advantages.

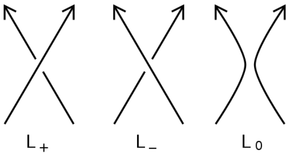
Focus

$\widehat{HF}$ homology, the most computable.

3-dimensional invariants

By contrast, there are many 3-manifold invariants.
Focus on a knot K in the space.

One of oldest: Alexander polynomial $\Delta(K)$.
Algebraic topology: Look at $H_1(\widetilde{\mathbb{R}^3 \setminus K})$ under deck transforms.
Skein theory: $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0)$



More recent: Jones polynomial, HOMFLYPT polynomial, . . .

How are 3- and 4-dimensional theories related?

Knot homologies

Many knot invariants are one- or two-variable Laurent polynomials.

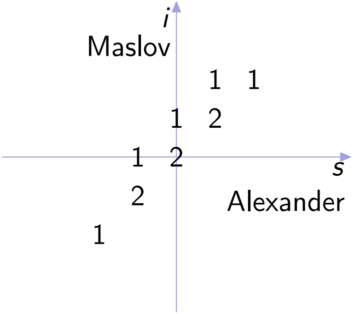
Can often find a doubly- or triply-graded homology theory whose Euler characteristic is the polynomial invariant.

Knot polynomial	Knot homology
Alexander (1928)	Heegaard Floer (2002) Instanton Floer (2008)
Jones (1983)	Khovanov (1999)
HOMFLYPT (1985)	Khovanov-Rozansky (2004)
Kauffman (1990)	Khovanov-Rozansky (2007) (conj)

Passage polynomial $\Rightarrow$ homology called *categorification*.

Heegaard Floer homology

$\dim(\widehat{HFK}_i(K;s))$:
$(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

- Bigraded;
- Euler characteristic is Conway-Alexander polynomial Δ;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
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We will give a simple algorithm for computing HFK...

...and so the world's simplest algorithm for knot genus!

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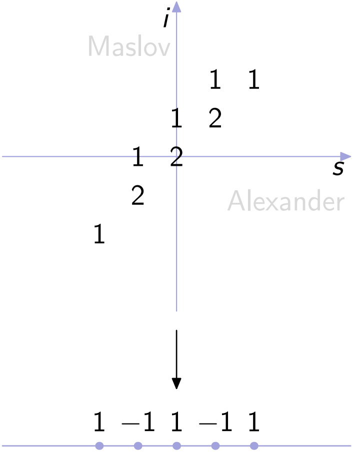

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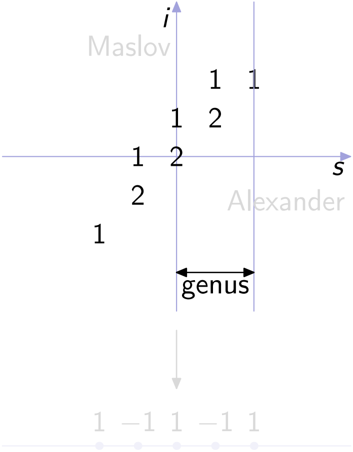

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Knot genus

Definition

A *Seifert surface* for a knot is an oriented surface embedded in space whose boundary is the knot.
The *genus* of a knot is the minimal genus of any Seifert surface.

Seifert surfaces always exist.
The genus of a knot is 0 iff it is the unknot.

Theorem (Neuwirth 1960)

Genus of $K \geq$ degree of Alexander-Conway polynomial

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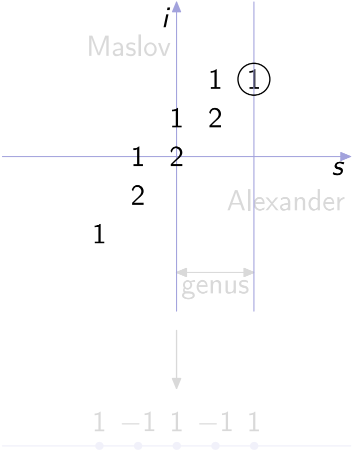

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Fibered knots

Definition

A knot is *fibered* if complement is a fiber bundle of a surface over the circle.
Or: Seifert surface can be swept around to cover complement.

Theorem (Neuwirth 1960)

K is fibered $\Rightarrow$ Alexander-Conway polynomial is monic

Theorem (Ghiggini-Ni 2006)

K is fibered $\Leftrightarrow \widehat{HFK}_i(K;s)$ is monic w.r.t. s
$\Leftrightarrow$ for max s so that $\widehat{HFK}_*(K;s) \neq 0$, $\dim(\widehat{HFK}_*(K;s)) = 1$

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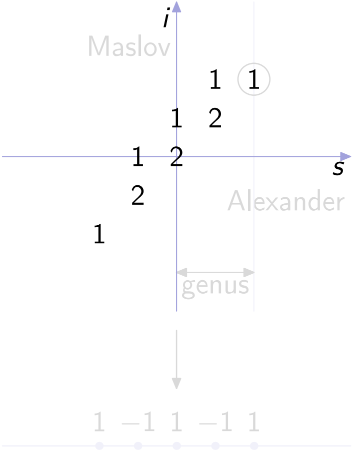

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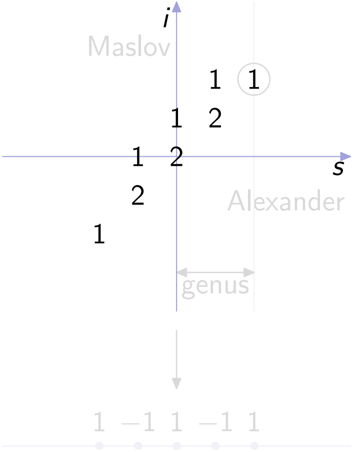

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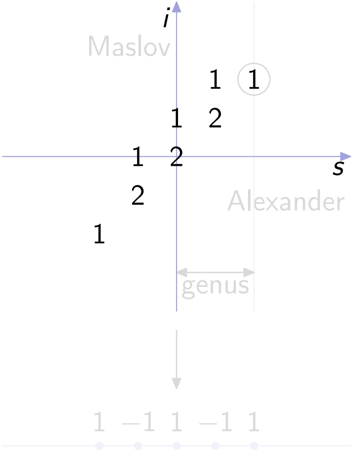

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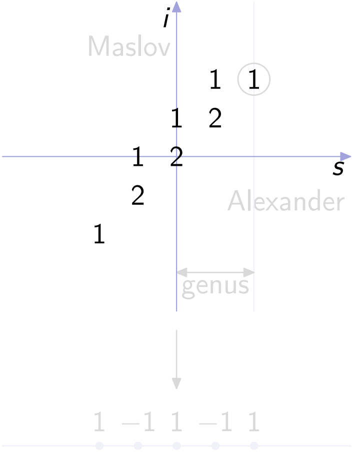

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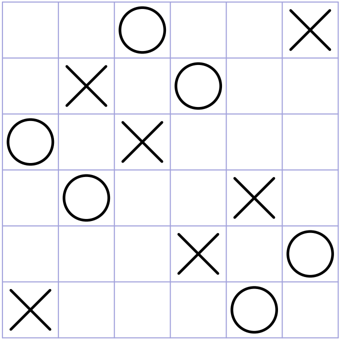

Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

O to X in each row.

Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:
Move top segment to bottom.

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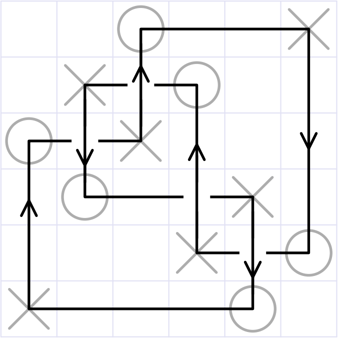

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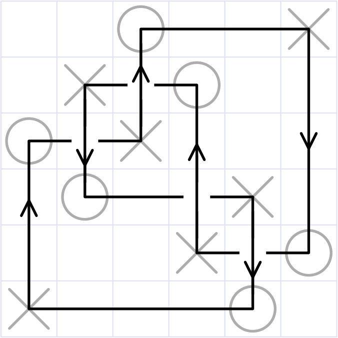

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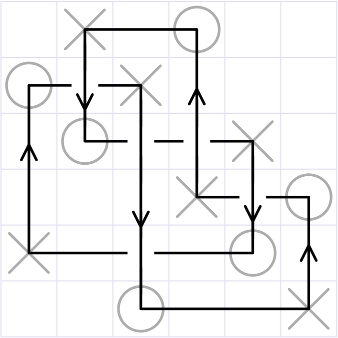

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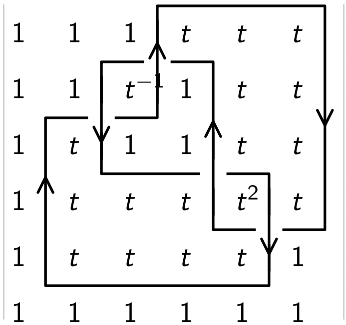

$$\begin{vmatrix} 1 & 1 & 1 & t & t & t \\ 1 & 1 & t^{-1} & 1 & t & t \\ 1 & t & 1 & 1 & t & t \\ 1 & t & t & t & t^2 & t \\ 1 & t & t & t & t & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \pm t^*(1-t)^{n-1}\Delta(K;t)$$

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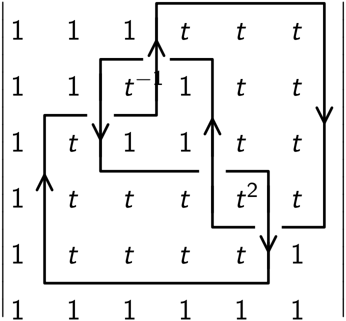

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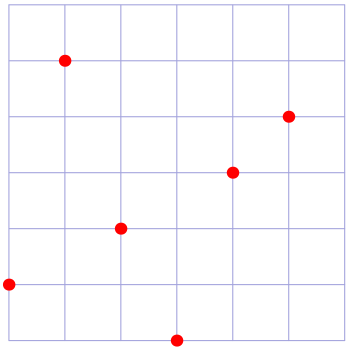
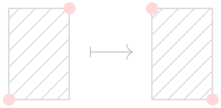
Computing HFK: Chain complex $\widetilde{CK}$



Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



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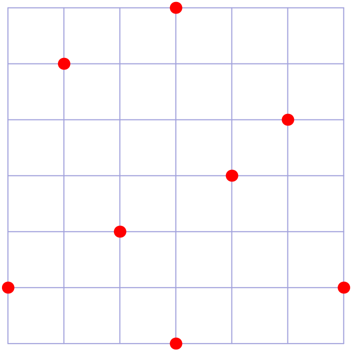

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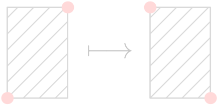

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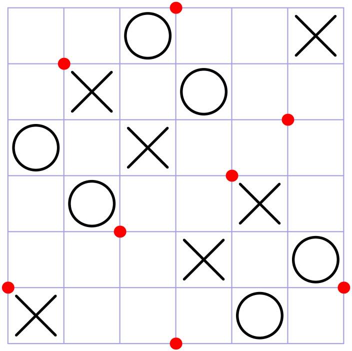

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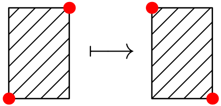

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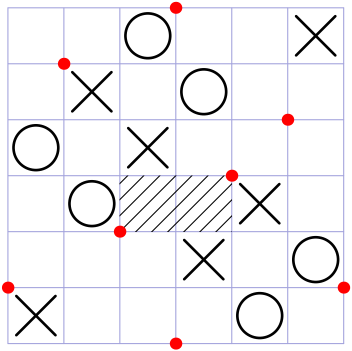

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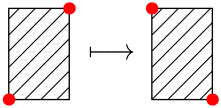

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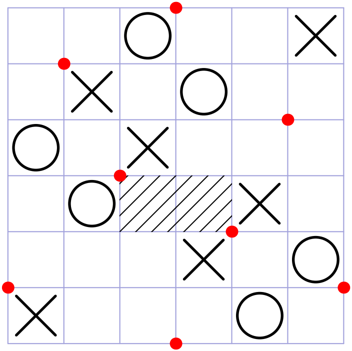

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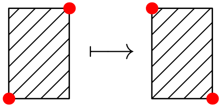

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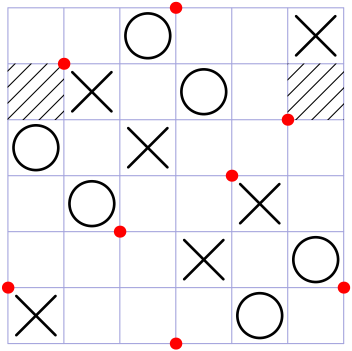

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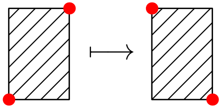

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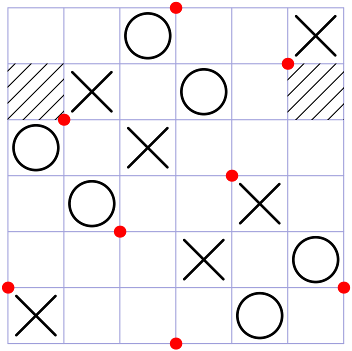

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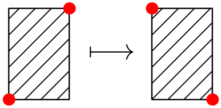

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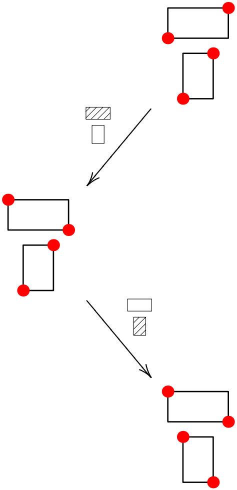

Each term in ∂^2 must have a mate:

- If rectangles are disjoint, take rectangles in either order.
- If rectangles share a corner, decompose the union in another way.

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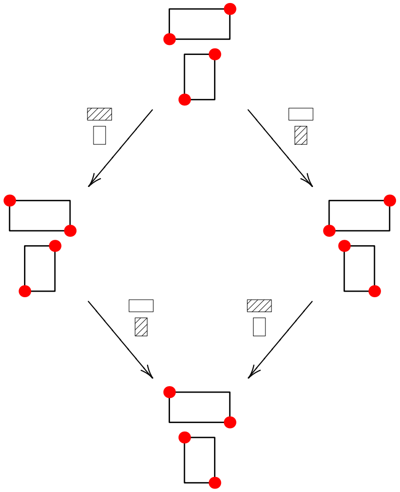

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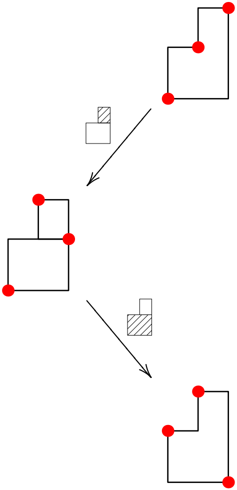

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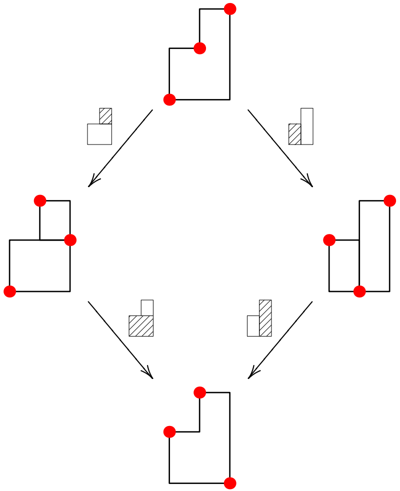

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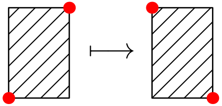
Computing *HFK*: Gradings on $\widetilde{CK}$

In the plane,



removes one *inversion*.

For $A, B, C \subset \mathbb{R}^2$,

$$\mathcal{I}(A, B) := \#\{ {}_a\Box^b \mid a \in A, b \in B \}$$
$$\mathcal{I}(A - B, C) := \mathcal{I}(A, C) - \mathcal{I}(B, C)$$

For $\mathbf{x}$ a generator, $\mathbb{X}$ = set of X's, $\mathbb{O}$ = set of of O's, gradings are:

- **Maslov:** $M(\mathbf{x}) := \mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1.$
- **Alexander:** Sum of winding numbers around generator pts, or $A(\mathbf{x}) := \frac{1}{2}\big(\mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) - \mathcal{I}(\mathbf{x} - \mathbb{X}, \mathbf{x} - \mathbb{X}) - (n-1)\big).$

Computing HFK: The answer

Theorem (Manolescu-Ozsváth-Sarkar)

For G a grid diagram for K,

$$H_*(\widetilde{CK}(G)) \simeq \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$.

(Remember the factor of $(1-t)^{n-1}$ in determinant formula for Δ.)

Gillam and Baldwin used this to compute $\widehat{HFK}$ for all knots with ≤ 11 crossings, including new values of knot genus.

Variants

- Allow rectangles to cross O's with a coefficient of U_i
  - Remove factor of $V^{\otimes n-1}$
  - Get stronger invariant HFK^- over $\mathbb{F}_2[U]$
- Allow rectangles to cross X's to get a filtered complex
- Add signs (in essentially unique way) to work over $\mathbb{Z}[U]$.

Cross O's? (count with U_i variables) \ Cross X's?	No	Yes
No	$\widehat{HFK} \otimes V^{n-1}$	Filtered version
All but 1	$\widehat{HFK}$	"
Yes	HFK^-	"

Outline

TQFT-like structure

Geometry	Algebra
Closed 4-manifold W^4, Spin^c structure $\mathfrak{s}$	Invariant $HF(W, \mathfrak{s})$
3-manifold Y^3, Spin^c structure $\mathfrak{s}$	Homology thy $HF(Y, \mathfrak{s})$
Cobordism $\partial W^4 = (-Y_1) \cup Y_2$	$HF(W) : HF(Y_1) \to HF(Y_2)$

Subtleties omitted here. E.g., if you make an ordinary TQFT, invariants of closed 4-manifolds are 0.

Similar story for knots in 3-manifolds with surface cobordisms.

We saw $HFK(S^3, K; \mathbb{F}_2)$, the easiest to understand.

Extending TQFT down

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
Surface F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CF}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CF}(Y_1) \widetilde{\otimes}_{\mathcal{A}(F)} \widehat{CF}(Y_2)$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CF}(Y), \ldots$
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$
Self-gluing	Hochschild (co)homology

This is the theory of Bordered Floer homology.

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Mapping class group

Special case: 3-manifold is $F \times [0,1]$, with two boundary components parametrized differently.

This corresponds to element of *mapping class group*: diffeomorphisms $\phi : F \to F$, considered up to isotopy.

Theorem (Lipshitz-Ozsváth-T)

There is an action of the strongly based mapping class group on modules over $\mathcal{A}(F)$.

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This action is faithful.

Outline

Further questions

There are many questions left!

- Combinatorial proofs of genus equality and other nice properties?
- Construct a topological space with this homology?
- Manifold version of other link homologies?
- Make entire theory computable?
- Do computations!

Appendix: Crossing number vs. Grid number

Knots are usually ordered by *crossing number*:
Minimum number of crossings in a planar diagram.

For grid diagrams, natural to consider *grid number* (or *arc index*):
Minimum size of a grid diagram.

Theorem (Bae–Park, Morton–Beltrami)

Grid number of an alternating knot is equal to crossing number $+2$.
For non-alternating knots, grid number strictly less.