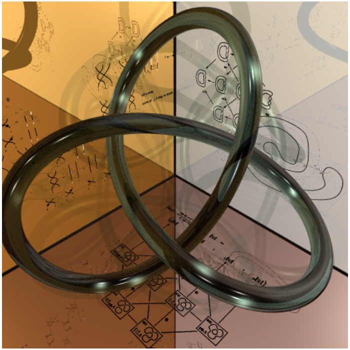
Combinatorial versions of Heegaard Floer homology

Dylan Thurston

http://www.math.columbia.edu/~dpt/speaking



November 26, 2012

Heegaard Floer homology

Goal

Compute Heegaard Floer homology for knots, 3-manifolds, and 4-manifolds, theoretically and in practice

Heeegaard-Floer homology (Ozsváth-Szabó 2000–2003) is (nearly) a 4D TQFT. It assigns

Geometry	Algebra
Smooth 4-manifold W^4	Invariant (number) $HF(W)$
3-manifold Y^3	Vect. space (homology) $HF(Y)$
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	$HF(W) : HF(Y_1) \to HF(Y_2)$
$W = W_1 \cup_Y W_2$	$HF(W) = HF(W_1) \circ HF(W_2)$

Caveats:

- Y_2 must be connected
- Need to work a little to get 4-manifold invariants
- 4-manifold invariants only defined if $b_2^+(W) \geq 2$

4-dimensional invariants

There are multiple smooth structures on the same topological 4-manifold.

Several theories give 4-manifold invariants to detect this:[1]

Donaldson theory
  $\leftrightarrow$ (conj. Seiberg–Witten '94)
Monopole Floer homology (Seiberg-Witten)
  $\cong$ (Taubes '08)
Embedded contact homology
  $\cong$ (Kutluhan-Lee-Taubes '11, Colin-Ghiggini-Honda '12)
Heegaard Floer homology

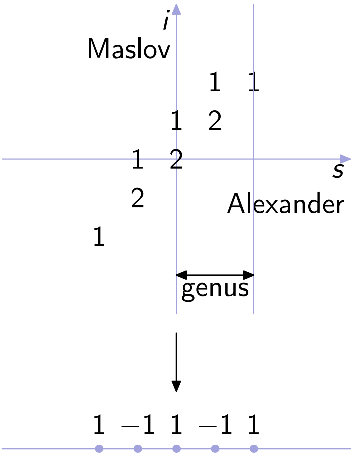
We topologists only have one trick in 4 dimensions!

Monopoles: More computable version of Donaldson invariants
HF homology: More computable version of monopoles

[1] Some theories/equivalences currently only work in 3 dimensions.

Heegaard Floer homology

$\dim(\widehat{HFK}_i(K;s))$:
$(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

- Bigraded;
- Euler characteristic is Conway-Alexander polynomial Δ;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

Knot genus

Definition

A *Seifert surface* for a knot is an oriented surface embedded in space whose boundary is the knot.
The *genus* of a knot is the minimal genus of any Seifert surface.

Seifert surfaces always exist.
The genus of a knot is 0 iff it is the unknot.

Theorem (Neuwirth 1960)

Genus of $K \geq$ degree of Alexander-Conway polynomial

Theorem (Ozsváth-Szabó 2001)

Genus of $K =$ max s so that $\widehat{HFK}_*(K; s) \neq 0$

Heegaard Floer homology

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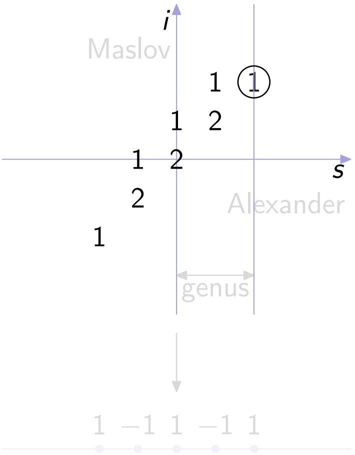

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Fibered knots

Definition

A knot is *fibered* if complement is a fiber bundle of a surface over the circle.
Or: Seifert surface can be swept around to cover complement.

Theorem (Neuwirth 1960)

K is fibered $\Rightarrow$ Alexander-Conway polynomial is monic

Theorem (Ghiggini-Ni 2006)

K is fibered $\Leftrightarrow \widehat{HFK}_i(K;s)$ is monic w.r.t. s
$\Leftrightarrow$ for max s so that $\widehat{HFK}_*(K;s) \neq 0$, $\dim(\widehat{HFK}_*(K;s)) = 1$

Knot homologies

More generally, many knot invariants are one- or two-variable Laurent polynomials.

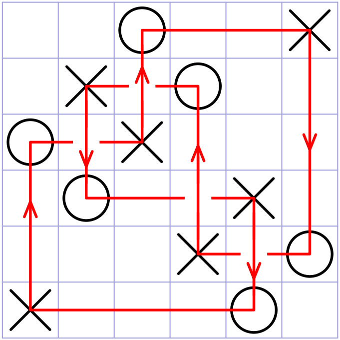
Can often find a doubly- or triply-graded homology theory whose Euler characteristic is the polynomial invariant.

Knot polynomial	Knot homology
Alexander (1928)	Heegaard Floer (2002), Ozsváth-Szabó Instanton Floer (2008), Kronheimer-Mrowka
Jones (1983)	Khovanov (1999)
HOMFLYPT (1985)	2004, Khovanov-Rozansky
Kauffman (1990)	2007, Khovanov-Rozansky (conj)

Passage polynomial $\Rightarrow$ homology called *categorification*.

Only in the Heegaard Floer case do we know how to extend to 3-manifolds.

Setting: Grid diagrams



Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

O to X in each row.

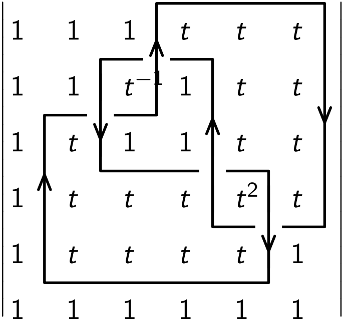
Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:
Move top segment to bottom.

Computing the Alexander polynomial

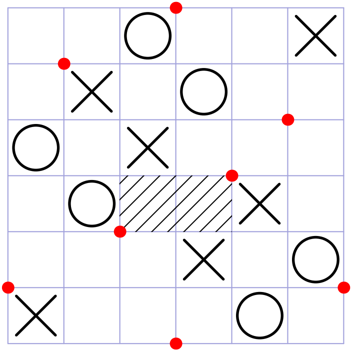
We categorify the following formula:



$$\begin{vmatrix} 1 & 1 & 1 & t & t & t \\ 1 & 1 & t^{-1} & 1 & t & t \\ 1 & t & 1 & 1 & t & t \\ 1 & t & t & t & t^2 & t \\ 1 & t & t & t & t & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \pm t^*(1-t)^{n-1}\Delta(K;t)$$

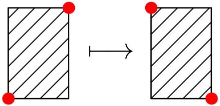
- Make matrix of $t^{-\text{winding }\#}$ (with extra row/column of 1's);
- det determines the Conway-Alexander polynomial Δ ($n =$ size of diagram; here 6)

Computing HFK: Chain complex $\widetilde{CK}$



Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

Computing HFK: The answer

Theorem (Manolescu-Ozsváth-Sarkar '06)

For G a grid diagram for K,

$$H_*(\widetilde{CK}(G)) \simeq \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$.

(Remember the factor of $(1-t)^{n-1}$ in determinant formula for Δ.)

Gillam and Baldwin used this to compute $\widehat{HFK}$ for all knots with ≤ 11 crossings, including new values of knot genus.

General structure of HF

Heegaard diagram $\mathcal{H}$: surface Σ with two sets of marked curves
$\boldsymbol{\alpha} = \bigcup \alpha_i$, $\boldsymbol{\beta} = \beta_i$
(No intersection within $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$)

Represents a 3-manifold:

- Take $\Sigma \times [0, 1]$
- Attach handles on $\boldsymbol{\alpha} \times \{0\}$, $\boldsymbol{\beta} \times \{1\}$
- Cap off boundaries

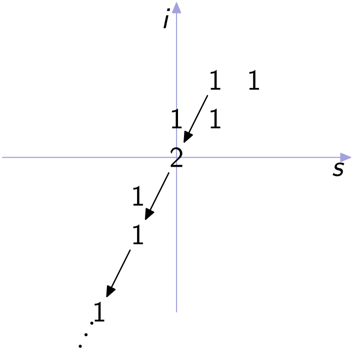
Generators: collections of points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ with

- One point on each α_i
- One point on each β_i

Differential: Count pseudo-holomorphic curves

The Heegaard Floer homology package

dim $HFK_i^-(K;s)$:



HF comes in many variants. So far, we have seen the simplest, $\widehat{HF}$. Also HF^+, HF^-, and HF^∞.

Thinking about variants will allow removing factors of $V^{\otimes n-1}$.

Also allows computing 4-manifold invariants.

Relation:

- CFK^- is a complex over $\mathbb{F}_2[U]$
- U has degree $(-1,-2)$
- Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set U to 0 on chains).

Computing HFK^-

To compute HFK^- for grid diagrams:

- Add one U_i for each O.
- Complex $CK^-(G)$ generated by same generators over $\mathbb{F}_2[U_1, \dots, U_n]$
- ∂ counts rects. that contain only O's, weighted by corresponding U_i.

Theorem (Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$.
Each U_i acts by U on the homology.

Other variants are similar, but treat O's in different ways.
E.g., CFK^∞ is similar, but over $\mathbb{F}_2[U, U^{-1}]$ instead of $\mathbb{F}_2[U]$.

Further variants

Can also:

- Allow rectangles to cross X's as well to get a filtered complex with trivial total homology.
- Add signs (in essentially unique way) to work over $\mathbb{Z}[U]$.

Cross O's? (count with U_i variables) \ Cross X's?	No	Yes
No	$\widehat{HFK} \otimes V^{n-1}$	Filtered version
All but 1	$\widehat{HFK}$	"
Yes	HFK^{-}	"

Method 1: Grid diagrams

Grid diagrams can compute Heegaard Floer homology for knots.
Works well for small examples.

Problem

- *Inefficient: basis of chain complex grows like* $n!$
- *Only for knots, not 3- or 4-manifolds*

Method 2: Nice diagrams

Definition

A *nice Heegaard diagram* is one in which all regions, except for one, have 2 or 4 sides.

Theorem (Sarkar-Wang '06)

Nice Heegaard diagrams exist for any 3-manifold.

In nice diagrams, holomorphic curve counts are easy: Count empty bigons, rectangles.

Problem

- *Nice diagrams are huge!*
- *Cannot handle 4-manifolds.*

Method 3: Surgery approach

Every 3-manifold is *surgery* on a link: cut out a neighborhood of the link and reglue another way.

For $K \subset S^3$ a knot, there is an exact triangle

$$\cdots \to HF(S^3) \to HF(S^3_0) \to HF(S^3_1) \to HF(S^3) \to \dots$$

S^3_0: Surgery on K. S^3_1: Surgery with different framing.

Manolescu-Ozsváth, Manolescu-Ozsváth-T: the effect of surgery can be computed using grid diagram techniques. Also compute 4-manifold invariants this way.

Problem

- *Complicated, inefficient.*

Method 4: Bordered Floer homology

Another approach: Cut 3-manifold into simpler pieces along surfaces. (Lipshitz-Ozsváth-T, ongoing)

Want: Extend HF as a TQFT down a dimension

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W, \mathfrak{s})$
3-manifold Y^3	Homology $HF(Y, \mathfrak{s})$
4-manifold w/ $\partial W^4 = Y$	$HF(W) \in HF(Y)$
Surface F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	Module $CF(Y)$ over $\mathcal{A}(F)$
$Y = Y_1 \cup_F Y_2$	$CF(Y) \simeq CF(Y_1) \otimes_{\mathcal{A}(F)} CF(Y_2)$

Method 4: Bordered Floer homology

Have: Extend HF as a TQFT down a dimension

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W, \mathfrak{s})$
3-manifold Y^3	Homology $HF(Y, \mathfrak{s})$
4-manifold w/ $\partial W^4 = Y$	$HF(W) \in HF(Y)$
Surface F	Differential algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	Differential, $\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Differential projective module $\widehat{CFD}(Y)$
$Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \widetilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$

Method 4: Bordered Floer homology

Benefits:

- Computability (theoretical)
- Computability (practical)
- Axioms

Notes:

- Two versions of modules, left and right actions
- Disconnected surfaces (etc.) behave differently
- Everything graded

Problem

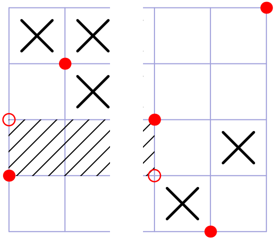
- *Only $\widehat{HF}$ so far, no closed 4-manifold invariants*

Splitting Heegaard diagrams

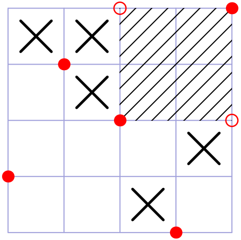
Recall that HF homology is defined based on Heegaard diagrams.

For bordered HF, split the diagram into two by stretching along a neck intersecting only α circles.

Need to keep track of differentials that cross the boundary.



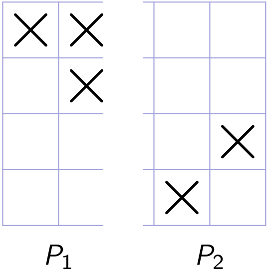
Toy model: Planar diagrams



A *grid diagram* represents a knot.
A *planar diagram* P is a square grid with blocks. Do not identify sides.
Chain complex $CF(P)$:

- Generators given by permutations
- Differential counts empty rectangles (*not* wrapping)
- Not an invariant of anything

Splitting planar diagrams



Want to split planar diagrams in two: $P = P_1 \cup P_2$.

Associate modules $CPA(P_1)$, $CPD(P_2)$

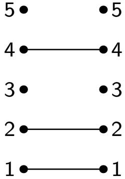
$$CP(P) = CPA(P_1) \otimes CPD(P_2)$$

$CPA(P_1)$ is a right differential module. Interactions on boundary encoded in algebra action.

$CPD(P_2)$ is a left, projective module. Interactions on boundary encoded in differential.

Tensor product is nice since $CPD(P_2)$ is projective.

Naive algebra $\widetilde{\mathcal{A}}(n,k)$

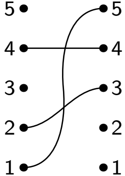


Idempotent
$\{1,2,4\} \subset \{1,\dots,5\}$

Defining a (naive) version of strands algebra $\widetilde{\mathcal{A}}(n,k)$ (really a category).

- Idempotents (objects): k-element subsets $S \subset \{1,\dots,n\}$
- Elements (morphisms): $\mathrm{Mor}(S,T)$ spanned by $\phi : S \overset{\sim}{\to} T$, $\phi(i) \geq i$
- Product: composition
- Differential: sum over smoothings of crossings

Naive algebra $\widetilde{\mathcal{A}}(n,k)$



Element
$1 \mapsto 5,\ 2 \mapsto 3,\ 4 \mapsto 4$

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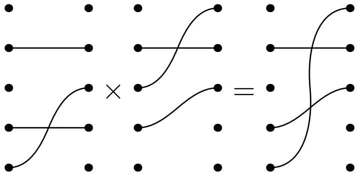

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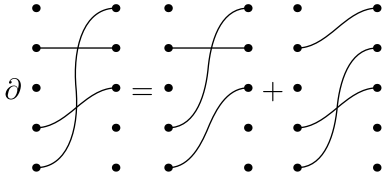

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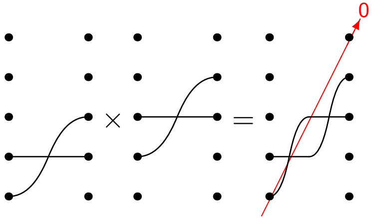
Strands algebra $\mathcal{A}(n, k)$

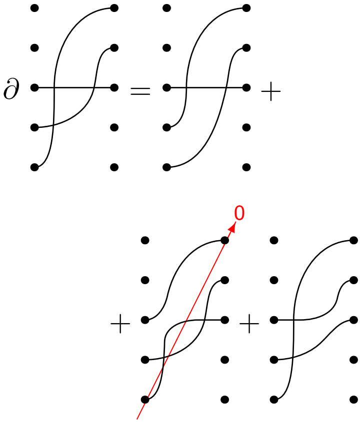


$\widetilde{\mathcal{A}}(n, k)$ has a filtration by number of crossings.

$\mathcal{A}(n, k)$ is the associated graded algebra.

- Product is 0 if it introduces a double-crossing.
- Differential: likewise.

Strands algebra $\mathcal{A}(n, k)$



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Torus algebra

For the torus, the answer is simple.

$$\mathcal{A}(T^2) = I_1 \underset{\rho_2}{\overset{\rho_1,\ \rho_3}{\rightleftarrows}} I_2 \Big/ (\rho_2\rho_1 = \rho_3\rho_2 = 0).$$

- Quiver algebra with relations
- Finite-dimensional
- No differential (special for torus)
- $\mathcal{A}(T^2)$ Koszul self-dual

Torus algebra for HF^-

Algebra $\mathcal{A}^-(T^2)$: $\mathcal{A}_\infty$ algebra

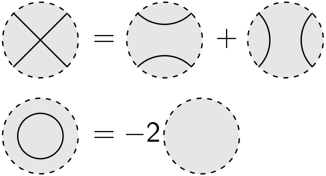
$$I_1 \underset{\rho_2,\ \rho_4}{\overset{\rho_1,\ \rho_3}{\rightleftarrows}} I_2 \Big/ (\rho_2\rho_1 = \rho_3\rho_2 = \rho_4\rho_3 = \rho_1\rho_4 = 0).$$

with additional elements ω, U satisfying

- $\omega^2 = 0$
- $d\omega = \rho_1\rho_2\rho_3\rho_4 + \rho_2\rho_3\rho_4\rho_1 + \rho_3\rho_4\rho_1\rho_2 + \rho_4\rho_1\rho_2\rho_3$
- $m_4(\rho_4, \rho_3, \rho_2, \rho_1) = U$ (and cyclic permutations)
- Higher products
- $\mathcal{A}^-(T^2)$ Koszul self-dual

Skein algebra for surfaces

The *skein algebra* $\mathrm{Sk}(\Sigma)$ for a surface Σ (at $q = 1$) is ambient isotopy classes of curves modulo the local relations

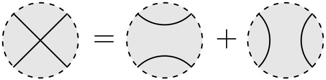


Union of curves gives algebra structure.

$\mathrm{Sk}(\Sigma)$ is the character variety of (twisted) SL_2 representations of $\pi_1(\Sigma)$. Compare: For $A, B \in SL_2$,

$$\mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}).$$

Strong positivity for skein algebra

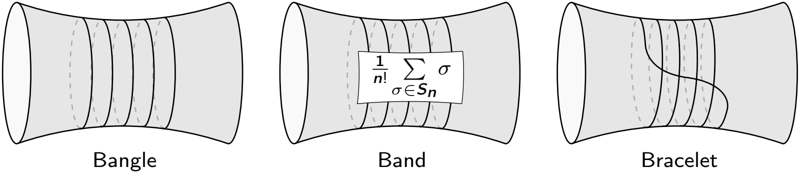


Theorem (T, 2012)

Sk(Σ) *has a natural, strongly positive basis.*

Natural: Invariant under automorphisms of Σ
Strongly positive: Structure constants for multiplication are positive

Basis is by *multi-bracelets*.



Categorification of skein algebra?

Conjecture

Sk(Σ) *has a monoidal categorification: There is a monoidal abelian category* $\mathcal{C}(\Sigma)$ *whose simple objects correspond to multi-bracelets, and which satisfies the skein relation in the Grothendieck ring.*

For instance, perhaps $\mathcal{C}(\Sigma)$ is the category of bimodules over some algebra; this is what happens in the bordered Floer case.

Positivity of coefficients suggests homological algebra is easier; e.g., modules rather than differential modules.

Problem

Find this monoidal categorification, and extract 4-manifold invariants from it.

Further questions

There are many questions left!

- Combinatorial proofs of genus equality and other nice properties?
- Construct a topological space with this homology?
- Manifold version of other link homologies?
- Make entire theory computable?
- Do computations!