Heegaard Floer Homology
Lecture 3: Structure of bordered HF homology

Dylan Thurston

Joint with Robert Lipshitz, Peter Ozsváth

`arXiv:0810.0687,1003.0598,1005.1248,???`
`http://www.math.columbia.edu/~dpt/speaking`

July 23, 2010, XIX OMGTP, Faro

Outline

- **Surfaces**
- Modules
- Bimodules
- Computability
- 4-manifolds

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
Surface F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y)$, ...
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
Parametrized surface F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y)$, ...
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Parametrized surfaces

A *handle decomposition* of a surface F is a way of writing F as

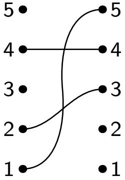
- a disk D_0,
- some handles attached to ∂D_0, and
- a disk D_2 attached to remaining boundary.

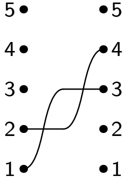
We also fix a basepoint on ∂D_0.

Definition

A *pointed matched circle* $\mathcal{Z}$ of genus g is a pairing of the points $\{1, \ldots, 4g\} \subset [0, 4g+1]$ so that surgery on pairs of matched points yields a connected interval.

Strands algebra $\mathcal{A}(n, k)$





$= 0$

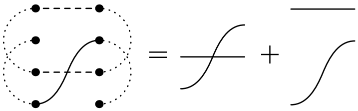
Recall strands algebra $\mathcal{A}(n, k)$ (really a category).

- Idempotents (objects): k-element subsets $S \subset \{1, \ldots, n\}$
- Elements (morphisms): $\mathrm{Mor}(S, T)$ spanned by $\phi : S \xrightarrow{\sim} T$, $\phi(i) \geqslant i$
- Product: composition
- Differential: sum over smoothings of crossings
- Set double-crossings to 0.

The algebra $\mathcal{A}(\mathcal{Z})$

For a pointed matched circle $\mathcal{Z}$ of genus g, we define $\mathcal{A}(\mathcal{Z}) \subset \mathcal{A}(4g) = \bigoplus_k \mathcal{A}(4g, k)$.

$\mathcal{A}(\mathcal{Z})$ is the subalgebra of sums of diagrams in which, if a diagram with a horizontal strand appears, the diagram with the horizontal strand at the matching position appears with equal weight.



The idempotents of $\mathcal{A}(\mathcal{Z})$ correspond to subsets of matched pairs (2^{2g} in all rather than 2^{4g}).

$\mathcal{A}(\mathcal{Z}, i)$ is the subalgebra with $g + i$ strands.

Properties of $\mathcal{A}(\mathcal{Z})$

- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\simeq \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

Properties of $\mathcal{A}(\mathcal{Z})$

- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

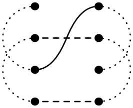
Properties of $\mathcal{A}(\mathcal{Z})$

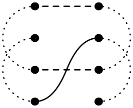
- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

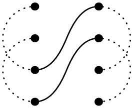
Properties of $\mathcal{A}(\mathcal{Z})$

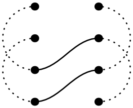
- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

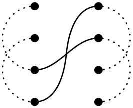
$\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$-graded

$x =$ 

$y =$ 

$x \cdot y =$ 

$\partial y =$ 

$(\partial y) \cdot x =$ 

$$x \cdot y = \partial((\partial y) \cdot x)$$

Aside: Non-commutative gradings

Let G by a group, possibly non-commutative.

An algebra A is G-graded if we have a decomposition

$$A = \bigoplus_{g \in G} A_g$$

so that

$$A_g \cdot A_h \subset A_{gh}.$$

For $\lambda \in G$ a fixed central element, a differential algebra is G-graded if in addition

$$\partial(A_g) \subset A_{\lambda^{-1}g}.$$

$\mathcal{A}(\mathcal{Z})$ is graded by $\begin{cases} \text{a canonical } \mathbb{Z} \text{ central extension of } H_1(F) \\ \text{vector fields on } F \times [0,1]/\text{isotopy rel } \partial. \end{cases}$

Properties of $\mathcal{A}(\mathcal{Z})$

- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

Properties of $\mathcal{A}(\mathcal{Z})$

- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

Properties of $\mathcal{A}(\mathcal{Z})$

- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

Properties of $\mathcal{A}(\mathcal{Z})$

- $\mathcal{A}(\mathcal{Z}, -g) \cong \mathbb{F}_2$.
- $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential, and is a quiver algebra.
- For $\mathcal{Z}$, $\mathcal{Z}'$ of same genus, $\mathcal{A}(\mathcal{Z}) \not\cong \mathcal{A}(\mathcal{Z}')$.
  Derived categories are isomorphic.
- $\mathcal{A}(\mathcal{Z})$ is not $\mathbb{Z}$ graded. It is G-graded for a non-commutative group G.
- $\mathcal{A}(\mathcal{Z}, i) \cong \mathcal{A}(-\mathcal{Z}, i)^{\mathrm{op}}$.
- $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to $\mathcal{A}(\mathcal{Z}, -i)$.
- $\mathcal{A}(\mathcal{Z}, i) \simeq \mathcal{A}(\mathcal{Z}^*, -i)$. ($\mathcal{Z}^*$ is the dual pointed matched circle.)
- $\mathcal{A}(\mathcal{Z})$ is derived equivalent to the "partially wrapped Fukaya category" of F [Auroux].

Outline

Surfaces

- **Modules**

Bimodules

Computability

4-manifolds

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y)$, ...
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Type D modules

Pointed matched circle $\mathcal{Z} \rightsquigarrow$ Surface $F(\mathcal{Z})$ with marked disk D_0

A *bordered 3-manifold* is a 3-manifold Y with a homeomorphism $\phi : F(\mathcal{Z}) \to \partial Y$, defined up to isotopy rel D_0.

Corresponding notion of bordered Heegaard diagrams $\mathcal{H}$, with boundary $\partial\mathcal{H}$ a pointed matched circle.

For $\mathcal{H}$ a bordered Heegaard diagram, $\widehat{CFD}(\mathcal{H})$ is a left, projective module over $\mathcal{A}(-\partial\mathcal{H}, 0)$.

Theorem

If $\mathcal{H}_1$ and $\mathcal{H}_2$ represent the same bordered 3-manifold,

$$\widehat{CFD}(\mathcal{H}_1) \simeq \widehat{CFD}(\mathcal{H}_2).$$

Can therefore write $\widehat{CFD}(Y)$ for Y a bordered 3-manifold.

Type A modules

For $\mathcal{H}$ a bordered Heegaard diagram, $\widehat{CFA}(\mathcal{H})$ is a right, $\mathcal{A}_\infty$ module over $\mathcal{A}(\partial\mathcal{H}, 0)$.

Theorem

If $\mathcal{H}_1$ and $\mathcal{H}_2$ represent the same bordered 3-manifold,

$$\widehat{CFA}(\mathcal{H}_1) \simeq \widehat{CFA}(\mathcal{H}_2).$$

A *nice* (based) Heegaard diagram in one in which every region (except for the one containing the basepoint) is a square or bigon.

Theorem (Sarkar-Wang)

Any 3-manifold has a nice diagram.

Lemma

If $\mathcal{H}$ is nice, then $\widehat{CFA}(\mathcal{H})$ is a differential module over $\mathcal{A}(\mathcal{Z})$.

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y), \ldots$
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
	$\widehat{CF}(Y) \simeq \mathrm{Hom}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2))$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y)$, ...
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Pairing theorems

Theorem

If Y_1 is bordered by $F(\mathcal{Z})$ and Y_2 is bordered by $-F(\mathcal{Z})$, then

$$\widehat{CF}(Y_1 \cup_\partial Y_2) \simeq \widehat{CFA}(Y_1) \,\widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})}\, \widehat{CFD}(Y_2).$$

Theorem

With Y_1, Y_2 as above,

$$\begin{aligned}\widehat{CF}(Y_1 \cup_\partial Y_2) &\simeq \mathrm{Mor}_{\mathcal{A}(\mathcal{Z})}(\widehat{CFA}(-Y_2), \widehat{CFA}(Y_1)) \\ &\simeq \mathrm{Mor}_{\mathcal{A}(-\mathcal{Z})}(\widehat{CFD}(-Y_2), \widehat{CFD}(Y_1))\end{aligned}$$

($\mathrm{Mor}(M, N)$ is a chain complex whose homology is $\mathrm{Ext}(M, N)$.)

Dualities

The two pairing theorems are related by dualities between $\widehat{CFA}$ and $\widehat{CFD}$.

Theorem

For Y bordered by $F(\mathcal{Z})$,

$$\begin{aligned} \mathrm{Mor}_{\mathcal{A}(-\mathcal{Z})}(\widehat{CFD}(Y), \mathcal{A}(-\mathcal{Z})) &\simeq \widehat{CFA}(-Y) \\ \mathrm{Mor}_{\mathcal{A}(\mathcal{Z})}(\widehat{CFA}(Y), \mathcal{A}(\mathcal{Z})) &\simeq \widehat{CFD}(-Y). \end{aligned}$$

Theorem (Suggested by Auroux)

For Y bordered by $F(\mathcal{Z})$,

$$\widehat{CFA}(Y, \mathfrak{s}) \simeq \widehat{CFD}(Y, \bar{\mathfrak{s}}).$$

Outline

- Surfaces
- Modules
- **Bimodules**
- Computability
- 4-manifolds

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
	$\widehat{CF}(Y) \simeq \mathrm{Hom}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2))$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y)$, ...
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Bimodules of various types

An *arced, bordered 3-manifold* is a 3-manifold Y with two boundary components, parametrized by $F(\mathcal{Z}_1)$ and $F(\mathcal{Z}_2)$, together with a framed arc connecting the base disks in the two boundary components.

For a Heegaard diagram $\mathcal{H}$ representing an arced, bordered 3-manifold, there are bimodules of various types:

$$\widehat{CFAA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_1,i),\mathcal{A}(\mathcal{Z}_2,-i)}$$
$$_{\mathcal{A}(-\mathcal{Z}_1,i)}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_2,i)}$$
$$_{\mathcal{A}(-\mathcal{Z}_1,i),\mathcal{A}(-\mathcal{Z}_2,-i)}\widehat{CFDD}(\mathcal{H}).$$

Theorem

For $\mathcal{H}$ representing an arced, bordered 3-manifold Y, the bimodules $\widehat{CFAA}(\mathcal{H})$, $\widehat{CFDA}(\mathcal{H})$, $\widehat{CFDD}(\mathcal{H})$ are invariants up to quasi-isomorphism of Y.

Pairing theorems

Theorem

We can glue (arced) bordered 3-manifold (bi)modules in any way that matches an A with a D.
For instance, if $\partial Y_1 = F(\mathcal{Z}_1) \cup F(\mathcal{Z}_2)$, $\partial Y_2 = -F(\mathcal{Z}_2)$,

$$\widehat{CFD}(Y_1 \cup_{F(\mathcal{Z}_2)} Y_2) = \widehat{CFDA}(Y_1) \otimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFD}(Y_2).$$

If $\partial Y_3 = -F(\mathcal{Z}_2) \cup F(\mathcal{Z}_3)$,

$$\widehat{CFDA}(Y_1 \cup_{F(\mathcal{Z}_2)} Y_3) = \widehat{CFDA}(Y_1) \otimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(Y_3).$$

There are also Hom-pairing and duality theorems for bimodules. Some of these involve a *boundary Dehn twist* τ_∂, which is the Serre functor in $\mathcal{A}(\mathcal{Z})$-Mod.

Theorem

$$\mathrm{Mor}_{\mathcal{A}(\mathcal{Z})}(N, M \otimes \widehat{CFDA}(\tau_\partial)) \simeq \mathrm{Mor}_{\mathcal{A}(\mathcal{Z})}(M, N)^*.$$

Mapping class group

As a special case, we can consider $Y = [0,1] \times F$, with the two boundaries possibly parametrized differently.

Theorem

$\widehat{CFDA}([0,1] \times F(\mathcal{Z})) \simeq \mathcal{A}(\mathcal{Z})$, with both boundaries parametrized by identity.

Corollary

There is a compositional map from the strongly based mapping class group of $F(\mathcal{Z})$ to $\mathcal{A}(\mathcal{Z})$ bimodules, so $MCG_0(F(\mathcal{Z}))$ acts (weakly) on $\mathcal{A}(\mathcal{Z})$-Mod.

Corollary

If $\mathcal{Z}$ and $\mathcal{Z}'$ have the same genus, $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z}')$ are derived equivalent.

Faithfulness of mapping class group action

Theorem

The mapping class group action of a surface of genus g on $\mathcal{A}(\mathcal{Z}, -g+1)$-Mod is faithful.

(Recall $\mathcal{A}(\mathcal{Z}, -g) = \mathbb{F}_2$, and $\mathcal{A}(\mathcal{Z}, -g+1)$ has no differential.)

Inspired by Seidel-Thomas '00: Rank of homology of bimodule counts intersections.

Conjecture

There is a faithful linear representation of the mapping class group.

Unfortunately, our representation presumably decategorifies to a relative for surfaces of Burau representation, which is not faithful.

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
	$\widehat{CF}(Y) \simeq \mathrm{Hom}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2))$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y), \dots$
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
	$\widehat{CF}(Y) \simeq \mathrm{Hom}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2))$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y), \dots$
Self-gluing	Hochschild (co)homology
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Hochschild (co)homology

Hochschild homology in a TQFT is typically related to self-gluing.
Idea:

$$HH_*({}_AM_A) = H_*(\mathrm{Tor}_{A\otimes A^{\mathrm{op}}}({}_AA_A, {}_AM_A)$$

For Y an arced, bordered 3-manifold param. by $\mathcal{Z}$ and $-\mathcal{Z}$. The *open book decomposition* (Y°, K) is obtained by gluing the two boundary components, doing surgery on framed knot coming from the framed arc in Y, retaining the core K of the surgery.

Theorem

For Y an arced, bordered 3-mfld as above,

$$HH_*(\widehat{CFDA}(Y)) \simeq \widehat{CFK}(Y^\circ, K).$$

Outline

- Surfaces
- Modules
- Bimodules
- **Computability**
- 4-manifolds

Computability

Everything is entirely computable! Especially the $\mathcal{A}_\infty$ modules.

$$\sum_i \dim(H_*(\mathcal{A}(\mathcal{Z}, i))) =$$

$$\begin{cases} T^{-2} + 32T^{-1} + 98 + 32T + T^2 & \mathcal{Z} \text{ split, genus 2} \\ T^{-2} + 32T^{-1} + 70 + 32T + T^2 & \mathcal{Z} \text{ antipodal, genus 2} \end{cases}$$

In a genus 3 example, get dimension 1224 in middle dimension.

Computing pairings by computing Hom-spaces multiplies by dimension of the algebra.

Computing pairing $\widehat{CFA} \otimes \widehat{CFD}$ has a model which does not increase rank at all.

For $\mathcal{A}_\infty$ modules allows passing to homology.

Computing in practice

Download a program-in-progress at
http://www.math.columbia.edu/~lipshitz/research.html.

Enough to find invariants of

- handlebodies and
- generators of mapping class group(oid).

This can be done.

Computations can be done in practice for genus 2.

Outline

- Surfaces
- Modules
- Bimodules
- Computability
- **4-manifolds**

Overview

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	$\mathcal{A}_\infty$ module $\widehat{CFA}(Y)$
	Projective module $\widehat{CFD}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$
	$\widehat{CF}(Y) \simeq \mathrm{Hom}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2))$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CFDA}(Y)$, ...
Self-gluing	Hochschild (co)homology
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Computing 4-manifold invariants

One approach to computing 4-manifold invariants: for T_0 and T_∞ the 0-framed and ∞-framed solid torus, respectively, compute a cobordism

$$\widehat{HF}(D^4) : \widehat{CFD}(T_0) \to \widehat{CFD}(T_\infty).$$

This works.

Easier approach: Use composition map

$$\mathrm{Mor}(\widehat{CFD}(Y_2), \widehat{CFD}(Y_3)) \otimes \mathrm{Mor}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2)) \\ \to \mathrm{Mor}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_3))$$

where Y_1, Y_2, Y_3 all parametrized by $\mathcal{Z}$. This is a map

$$\widehat{CF}(-Y_2 \cup_\partial Y_3) \otimes \widehat{CF}(-Y_1 \cup_\partial Y_2) \to \widehat{CF}(-Y_1 \cup_\partial Y_3).$$

Theorem

This is the cobordism map for a pair-of-pants cobordism.