Heegaard Floer Homology
Lecture 1: HF Homology from Grid Diagrams

Dylan Thurston

http://www.math.columbia.edu/~dpt/speaking

July 19, 2010, XIX OMGTP, Faro

Outline

- **Heegaard Floer homology in context**
- Grid diagrams
- Computing $\widehat{HFK}$
- General structure of HF homology

Many equivalent theories

Several theories giving 4-manifold invariants:

Donaldson theory
$\leftrightarrow$ (conj. Seiberg–Witten '94)
Monopoles (Seiberg-Witten)
$\cong$ (Taubes '08)
Embedded contact homology (ECH)
$\cong$ (Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda '10)
Heegaard Floer (HF) homology

At least for 3-mfld invariants

We topologists only have one trick!

Each theory has advantages.

Focus

$\widehat{HF}$ homology, the most computable.

TQFT-like structure

Geometry	Algebra
Closed 4-manifold W^4, Spin^c structure $\mathfrak{s}$	Invariant $HF(W,\mathfrak{s})$
3-manifold Y^3, Spin^c structure $\mathfrak{s}$	Homology thy $HF(Y,\mathfrak{s})$
Cobordism $\partial W^4 = (-Y_1) \cup Y_2$	$HF(W) : HF(Y_1) \to HF(Y_2)$

Subtleties omitted here. E.g., if you make an ordinary TQFT, invariants of closed 4-manifolds are 0.

Similar story for knots in 3-manifolds with surface cobordisms.

Focus

$HFK(S^3, K; \mathbb{F}_2)$, *the easiest to understand*

Knot homologies in general

Many knot invariants are one- or two-variable Laurent polynomials, associated to quantum groups.

Can often find a doubly- or triply-graded homology theory whose Euler characteristic is the polynomial invariant.

Group	Knot poly	Knot homology
$SL(2)$	Jones $J(t)$	Khovanov (1999)
$SL(n)$	HOMFLY $H(a, z)$	Kh-Roz (2004) ($n \in \mathbb{Z}$) Kh-Roz (2005) (n variable)
$GL(1 \mid 1)$	Alexander $\Delta(t)$	Heegaard Floer Seiberg–Witten Floer
$OSp(n)$	Kauffman $F(a, z)$	Kh-Roz (2007) (conjectural)

Categorification is passage polynomial $\Rightarrow$ homology.

Properties of $\widehat{HFK}$

$\dim(\widehat{HFK}_i(K; s))$: $(K = 10_{132})$

Characteristics of $\widehat{HFK}$:

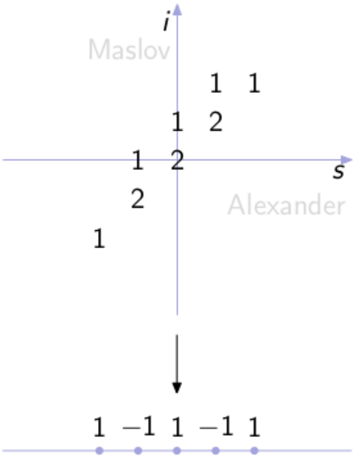
- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing $\widehat{HFK}$...

...and so the world's simplest algorithm for knot genus!

Properties of $\widehat{HFK}$

$\dim(\widehat{HFK}_i(K;s))$: $(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

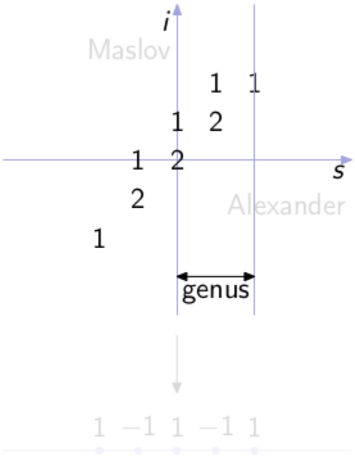
- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing $\widehat{HFK}$...

...and so the world's simplest algorithm for knot genus!

Properties of $\widehat{HFK}$

$\dim(\widehat{HFK}_i(K;s))$: $(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

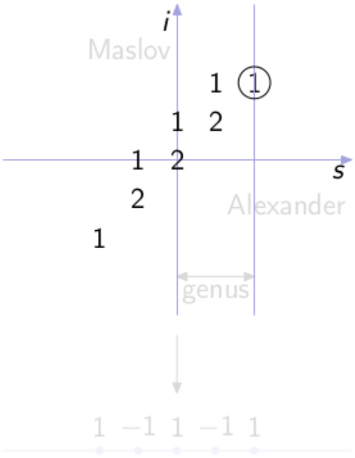
- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing *HFK*. . .

. . . and so the world's simplest algorithm for knot genus!

Properties of $\widehat{HFK}$

$\dim(\widehat{HFK}_i(K;s))$: $(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

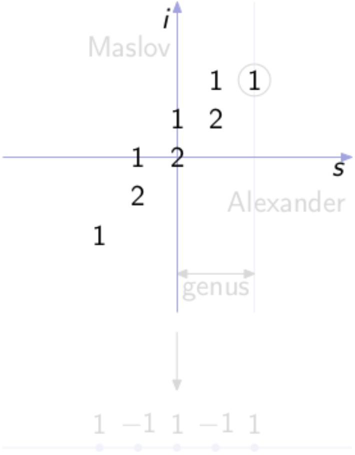
- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing HFK. . .

. . . and so the world's simplest algorithm for knot genus!

Properties of $\widehat{HFK}$

$\dim(\widehat{HFK}_i(K;s))$: $(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

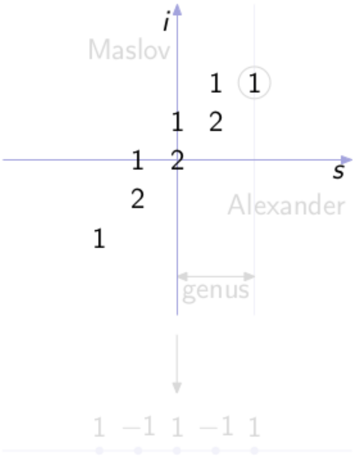
- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing $\widehat{HFK}$...

...and so the world's simplest algorithm for knot genus!

Properties of $\widehat{HFK}$

$\dim(\widehat{HFK}_i(K;s))$: $(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

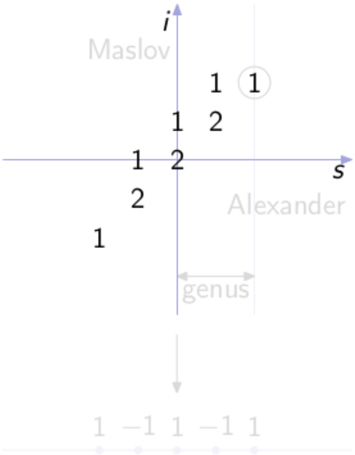
- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing $\widehat{HFK}$...

...and so the world's simplest algorithm for knot genus!

Properties of $\widehat{HFK}$

$\dim(\widehat{HFK}_i(K;s))$: $(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing HFK. . .

. . . and so the world's simplest algorithm for knot genus!

Outline

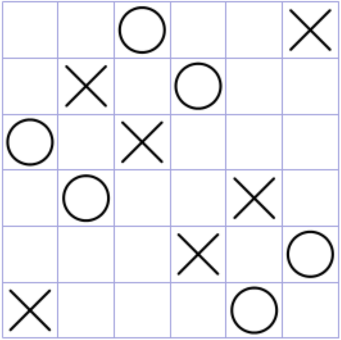
Heegaard Floer homology in context

- **Grid diagrams**

Computing $\widehat{HFK}$

General structure of HF homology

Setting: Grid diagrams



Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

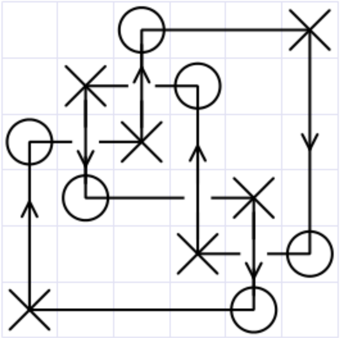
O to X in each row.

Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:
Move top segment to bottom.
Think about diagram on a torus.

Setting: Grid diagrams



Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

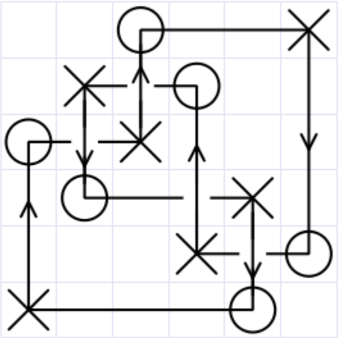
O to X in each row.

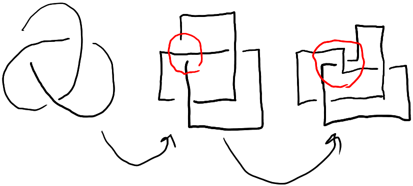
Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:
Move top segment to bottom.
Think about diagram on a torus.

Setting: Grid diagrams





Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

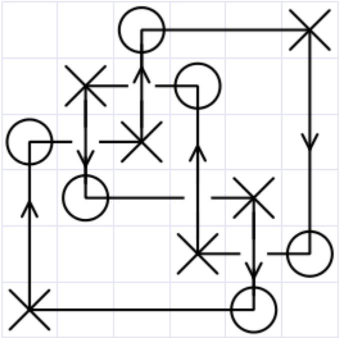
O to X in each row.

Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:
Move top segment to bottom.
Think about diagram on a torus.

Setting: Grid diagrams



Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

O to X in each row.

Cross vertical strands over horizontal.

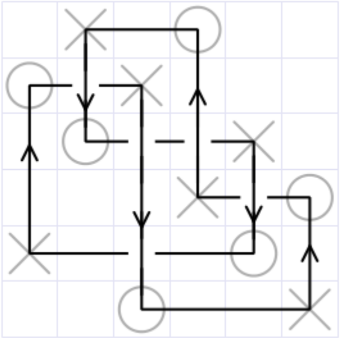
Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:

Move top segment to bottom.

Think about diagram on a torus.

Setting: Grid diagrams



Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

O to X in each row.

Cross vertical strands over horizontal.

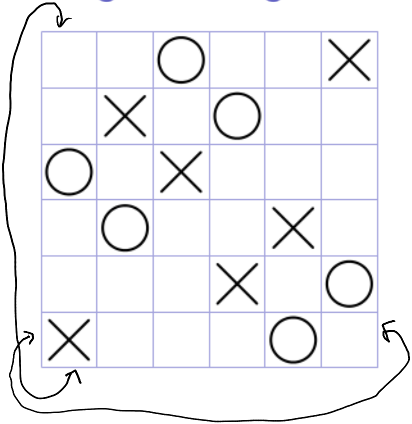
Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

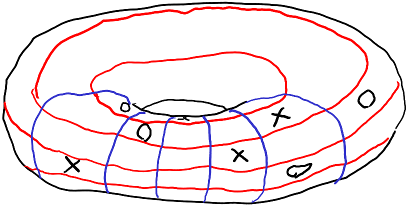
The knot is unchanged under *cyclic rotations*:

Move top segment to bottom.

Think about diagram on a torus.

Setting: Grid diagrams





Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

O to X in each row.

Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:
Move top segment to bottom.
Think about diagram on a torus.

Computing the Alexander polynomial

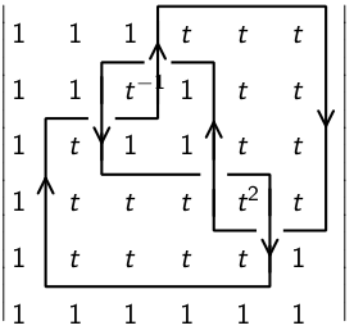
We categorify the following formula:

$$\begin{vmatrix} 1 & 1 & 1 & t & t & t \\ 1 & 1 & t^{-1} & 1 & t & t \\ 1 & t & 1 & 1 & t & t \\ 1 & t & t & t & t^2 & t \\ 1 & t & t & t & t & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \pm t^{*}(1-t)^{n-1}\Delta(K;t)$$

- Make matrix of $t^{-\text{winding }\#}$ (with extra row/column of 1's);
- det determines the Conway-Alexander polynomial Δ (n = size of diagram; here 6)

Computing the Alexander polynomial

We categorify the following formula:



$$\begin{vmatrix} 1 & 1 & 1 & t & t & t \\ 1 & 1 & t^{-1} & 1 & t & t \\ 1 & t & 1 & 1 & t & t \\ 1 & t & t & t & t^2 & t \\ 1 & t & t & t & t & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \pm t^*(1-t)^{n-1}\Delta(K;t)$$

- Make matrix of $t^{-\text{winding \#}}$ (with extra row/column of 1's);
- det determines the Conway-Alexander polynomial Δ (n = size of diagram; here 6)

Outline

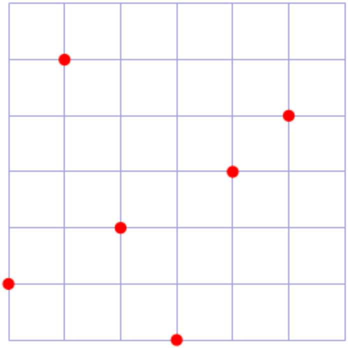
Heegaard Floer homology in context

Grid diagrams

- **Computing $\widehat{HFK}$**

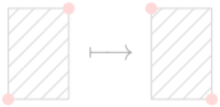
General structure of HF homology

Computing $\widehat{HFK}$: Chain complex $\widetilde{CK}$



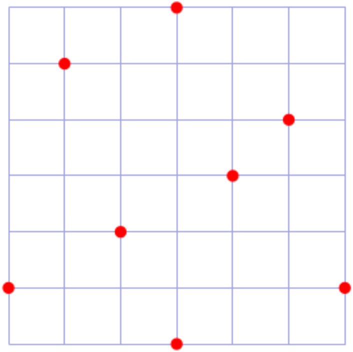
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



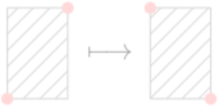
Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

Computing $\widehat{HFK}$: Chain complex $\widetilde{CK}$



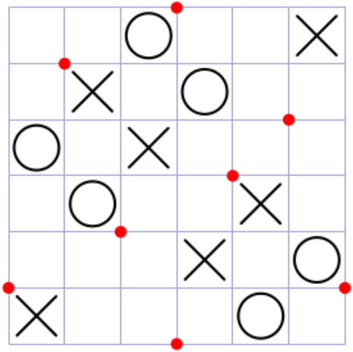
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



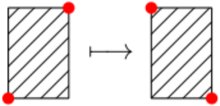
Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

Computing $\widehat{HFK}$: Chain complex $\widetilde{CK}$



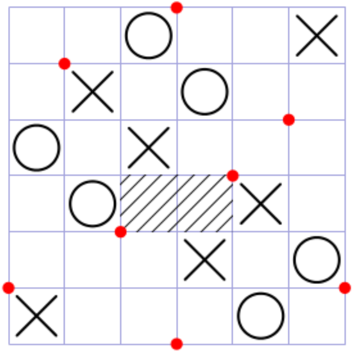
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



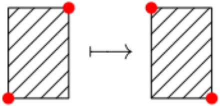
Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

Computing $\widehat{HFK}$: Chain complex $\widetilde{CK}$



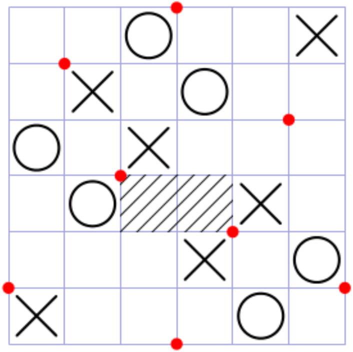
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



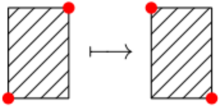
Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

Computing $\widehat{HFK}$: Chain complex $\widetilde{CK}$



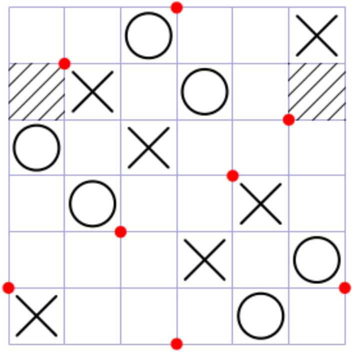
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



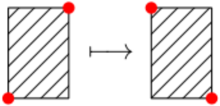
Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

Computing $\widehat{HFK}$: Chain complex $\widetilde{CK}$



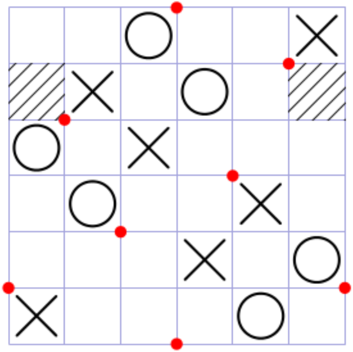
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



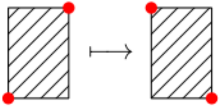
Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

Computing $\widehat{HFK}$: Chain complex $\widetilde{CK}$



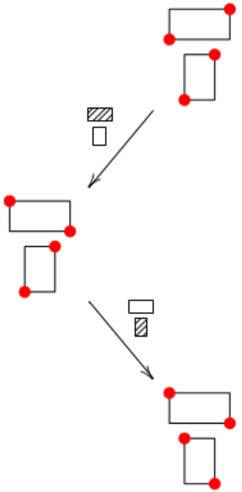
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

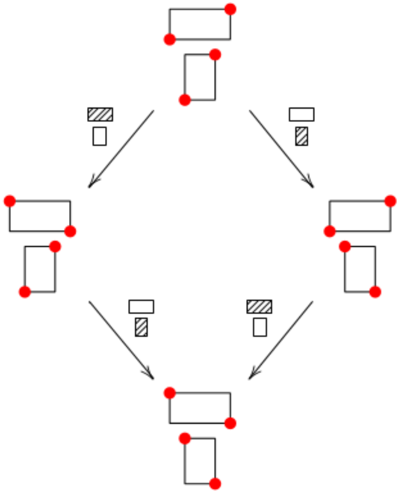
Computing $\widehat{HFK}$: $\partial^2 = 0$



Each term in ∂^2 must have a mate:

- If rectangles are disjoint, take rectangles in either order.
- If rectangles share a corner, decompose the union in another way.

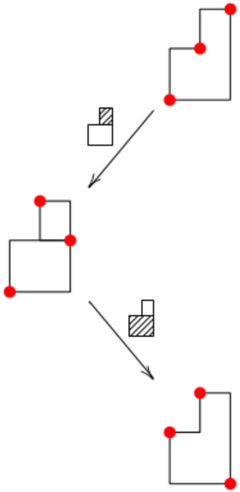
Computing $\widehat{HFK}$: $\partial^2 = 0$



Each term in ∂^2 must have a mate:

- If rectangles are disjoint, take rectangles in either order.
- If rectangles share a corner, decompose the union in another way.

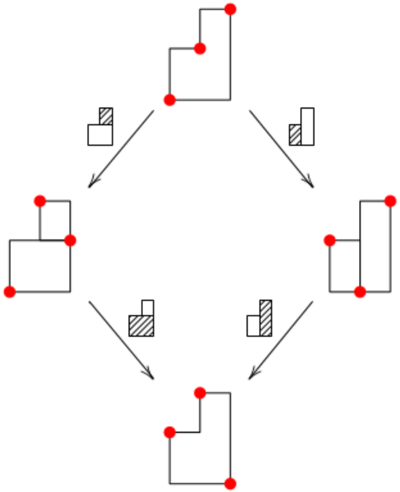
Computing $\widehat{HFK}$: $\partial^2 = 0$



Each term in ∂^2 must have a mate:

- If rectangles are disjoint, take rectangles in either order.
- If rectangles share a corner, decompose the union in another way.

Computing $\widehat{HFK}$: $\partial^2 = 0$

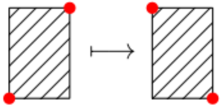


Each term in ∂^2 must have a mate:

- If rectangles are disjoint, take rectangles in either order.
- If rectangles share a corner, decompose the union in another way.

Computing $\widehat{HFK}$: Gradings on $\widetilde{CK}$

In the plane,



removes one *inversion*.

For $A, B, C \subset \mathbb{R}^2$,

$$\mathcal{I}(A,B) := \#\{\, {}_a\square^b \mid a \in A, b \in B \,\}$$
$$\mathcal{I}(A-B,C) := \mathcal{I}(A,C) - \mathcal{I}(B,C)$$

For $\mathbf{x}$ a generator, $\mathbb{X}$ = set of X's, $\mathbb{O}$ = set of of O's, gradings are:

- **Maslov:** $M(\mathbf{x}) := \mathcal{I}(\mathbf{x}-\mathbb{O}, \mathbf{x}-\mathbb{O}) + 1$.
- **Alexander:** Sum of winding numbers around generator pts, or $A(\mathbf{x}) := \frac{1}{2}\big(\mathcal{I}(\mathbf{x}-\mathbb{O}, \mathbf{x}-\mathbb{O}) - \mathcal{I}(\mathbf{x}-\mathbb{X}, \mathbf{x}-\mathbb{X}) - (n-1)\big)$.

Computing $\widehat{HFK}$: The answer

Theorem (Manolescu-Ozsváth-Sarkar)

For G a grid diagram for K,

$$H_*(\widetilde{CK}(G)) \simeq \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$.

(Remember the factor of $(1-t)^{n-1}$ in determinant formula for Δ.)

Gillam and Baldwin used this to compute $\widehat{HFK}$ for all knots with ≤ 11 crossings, including new values of knot genus.

Outline

Heegaard Floer homology in context

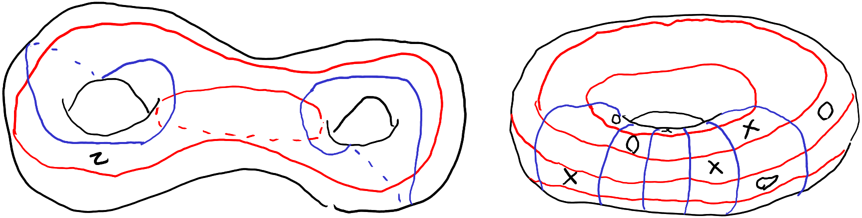
Grid diagrams

Computing $\widehat{HFK}$

- **General structure of HF homology**

Heegaard diagrams

Heegaard diagram $\mathcal{H}$: surface Σ with two sets of marked curves
$\boldsymbol{\alpha} = \bigcup \alpha_i$, $\boldsymbol{\beta} = \beta_i$
(No intersection within $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$)



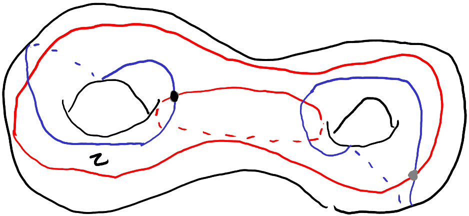
Represents a 3-manifold:

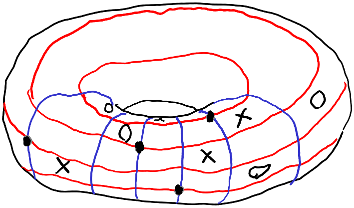
- Take $\Sigma \times [0, 1]$
- Attach handles on $\boldsymbol{\alpha} \times \{0\}$, $\boldsymbol{\beta} \times \{1\}$
- Cap off boundaries

Complex $CF(\mathcal{H})$

Generators: collections of points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ with

- One point on each α_i
- One point on each β_i



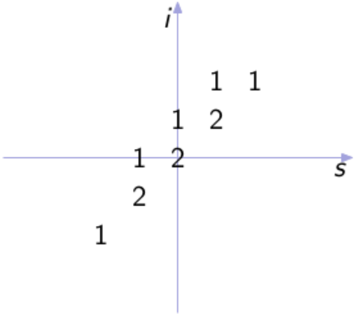


Differential: Count pseudo-holomorphic curves

- In $\mathrm{Sym}^k(\Sigma)$ (Ozsváth-Szabó)
- In $\Sigma \times [0, 1] \times \mathbb{R}$ (Cylindrical, Lipshitz)

Another variant: HF^-

$\dim \widehat{HFK}_i(K; s)$:



To remove factors of $V^{\otimes n-1}$:

Complex HFK^-: variant of $\widehat{HFK}$
Module over $\mathbb{F}_2[U]$
U has degree $(-1, -2)$
Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set U to 0 on chains).

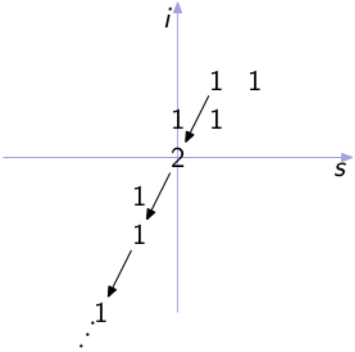
To compute: Add one U_i for each O.
Complex $CK^-(G)$ over $\mathbb{F}_2[U_1, \ldots, U_n]$
∂ counts rects. that contain only O's, weighted by corresponding U_i.

Theorem (Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$.
Each U_i acts by U on the homology.

Another variant: HF^-

dim $HFK^-_i(K;s)$:



To remove factors of $V^{\otimes n-1}$:

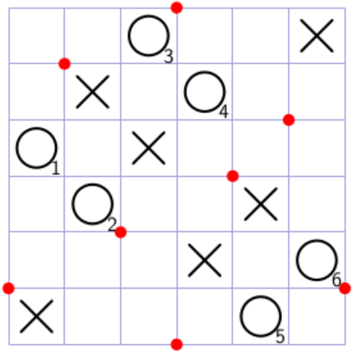
Complex HFK^-: variant of $\widehat{HFK}$
Module over $\mathbb{F}_2[U]$
U has degree $(-1,-2)$
Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set U to 0 on chains).

To compute: Add one U_i for each O.
Complex $CK^-(G)$ over $\mathbb{F}_2[U_1,\ldots,U_n]$
∂ counts rects. that contain only O's, weighted by corresponding U_i.

Theorem (Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$.
Each U_i acts by U on the homology.

Another variant: HF^-



To remove factors of $V^{\otimes n-1}$:

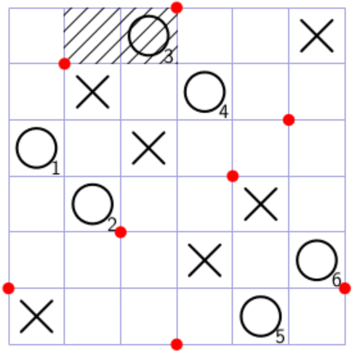
Complex HFK^-: variant of $\widehat{HFK}$
Module over $\mathbb{F}_2[U]$
U has degree $(-1, -2)$
Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set U to 0 on chains).

To compute: Add one U_i for each O.
Complex $CK^-(G)$ over $\mathbb{F}_2[U_1, \ldots, U_n]$
∂ counts rects. that contain only O's, weighted by corresponding U_i.

Theorem (Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$.
Each U_i acts by U on the homology.

Another variant: HF^-



To remove factors of $V^{\otimes n-1}$:

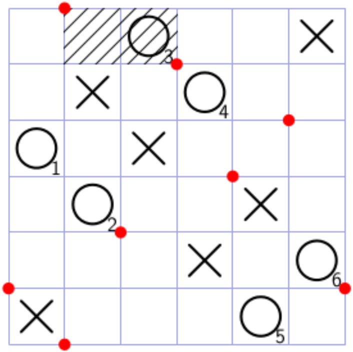
Complex HFK^-: variant of $\widehat{HFK}$
Module over $\mathbb{F}_2[U]$
U has degree $(-1,-2)$
Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set U to 0 on chains).

To compute: Add one U_i for each O.
Complex $CK^-(G)$ over $\mathbb{F}_2[U_1,\dots,U_n]$
∂ counts rects. that contain only O's, weighted by corresponding U_i.

Theorem (Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$.
Each U_i acts by U on the homology.

Another variant: HF^-



To remove factors of $V^{\otimes n-1}$:

Complex HFK^-: variant of $\widehat{HFK}$
Module over $\mathbb{F}_2[U]$
U has degree $(-1,-2)$
Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set U to 0 on chains).

To compute: Add one U_i for each O.
Complex $CK^-(G)$ over $\mathbb{F}_2[U_1,\dots,U_n]$
∂ counts rects. that contain only O's, weighted by corresponding U_i.

Theorem (Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$.
Each U_i acts by U on the homology.

Flavors of HF

In general, count curves S with a coefficient of $U^{n_z(S)}$
$n_z(S) = \#$ of times S covers basepoint z

Coefficients in	yields
$\mathbb{F}_2[U]/(U=0)$	$\widehat{HF}(Y)$
$\mathbb{F}_2[U]$	$HF^-(Y)$
$\mathbb{F}_2[U,U^{-1}]$	$HF^\infty(Y)$ (determined by $H^*(Y)$)
$\mathbb{F}_2[U,U^{-1}]/\mathbb{F}_2[U]$	$HF^+(Y)$

Compare: Equivariant cohomology, homology of total space, homology of fixed set

Exact triangle:

$$\cdots \to HF^-(Y) \to HF^\infty(Y) \to HF^+(Y) \to \dots$$

4-manifold invariants: The problem

Geometry	Algebra
3-manifold Y^3, Spin^c structure $\mathfrak{s}$	Homology thy $HF(Y, \mathfrak{s})$
Cobordism $\partial W^4 = (-Y_1) \cup Y_2$	$HF(W) : HF(Y_1) \to HF(Y_2)$

Works for any of $\widehat{HF}$, HF^-, HF^∞, HF^+

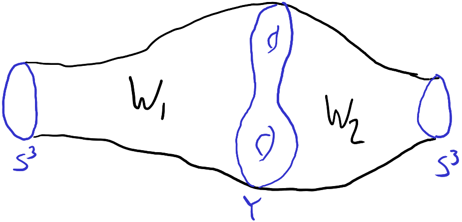
Usual way to get invariant of W^4: cobordism from S^3 to S^3
Get zero!

4-manifold invariants: The solution

Lemma

If W^4 is a cobordism with $b_2^+(W) > 0$, then $HF^\infty(W) = 0$ as map.

For $b_2^+(W) \geq 2$, split $W = W_1 \cup_Y W_2$



$$
\begin{array}{ccccc}
 & & HF^\infty(Y) & & \\
 & & \downarrow & & \\
 & & HF^+(Y) & \xrightarrow{HF^+(W_2)} & HF^+(S^3) \\
 & & \downarrow & & \\
HF^-(S^3) & \xrightarrow{HF^-(W_1)} & HF^-(Y) & & \\
 & & \downarrow & & \\
 & & HF^\infty(Y) & &
\end{array}
$$

Appendix: Crossing number vs. Grid number

Knots are usually ordered by *crossing number*:
Minimum number of crossings in a planar diagram.

For grid diagrams, natural to consider *grid number* (or *arc index*):
Minimum size of a grid diagram.

Theorem (Bae–Park, Morton–Beltrami)

Grid number of an alternating knot is equal to crossing number $+2$.
For non-alternating knots, grid number strictly less.