On canonical bases for surface cluster algebras

Have constructed:

1) Cluster algebra associated to surface (S, M)

2) Realization of (1) inside "coordinate ring" of Teichmüller space (with decorations)

(2) was motivation, but proof of (1) independent of (2) (in most cases)

Today

I. Does geometry help give more info?

II. Non-simply-laced case

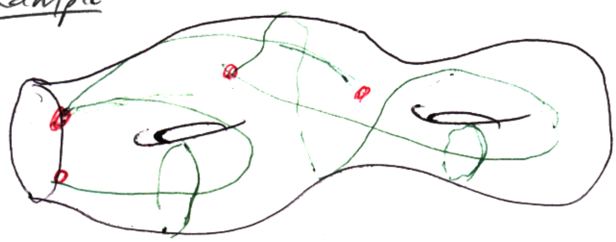
Curves basis

- Linear basis for ring containing cluster variables
- Described by Fock-Goncharov (and generalized)
- Positive? Yes Canonical? No

Def A multi-curve on (S, M) is a collection of circles and intervals mapped into S so that

- endpoints map to M
- interior of intervals, curves miss M, bdy. of S

Example



Note Intersections allowed! If no intersections, called simple.

Def The λ-length of a multi-curve C w.r.t. a decorated hyp. metric $\Sigma \in \tilde{\mathcal{T}}(S, M)$ is:

- If C is a interval mapped to S, $\lambda_\Sigma(C) = e^{\ell_\Sigma(C)/2}$, measured between horocycles
- If C is a circle mapped to S, $\lambda_\Sigma(C) = e^{\ell_\Sigma(C)/2} + e^{-\ell_\Sigma(C)/2}$

length of geodesic representative

- If $C = \bigcup_{i=1}^{k} C_i$, $\lambda(C) = \prod_{i=1}^{k} \lambda(C_i)$.

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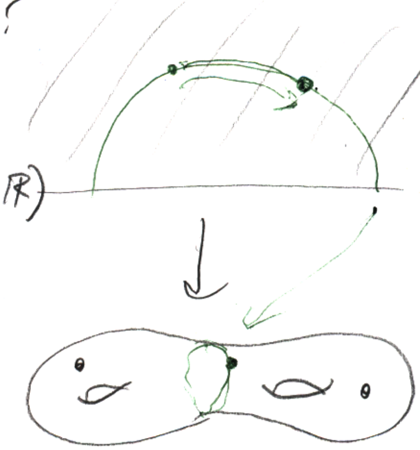
For circles, why is $\lambda(C)$ defined as above?

Consider lift $\tilde{C}$ of C to $\tilde{\Sigma} \subset \mathbb{H}$.



Covering transformation $\rightsquigarrow$ elt. $g(C) \in PSL(2,\mathbb{R})$

Lift to $M(C) \in SL(2,\mathbb{R})$

Concretely, $M(C)$ is conjugate to

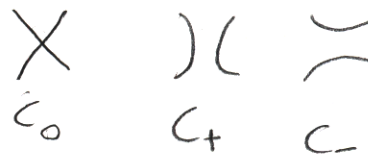
$$\pm \begin{pmatrix} e^{\ell(C)/2} & 0 \\ 0 & e^{-\ell(C)/2} \end{pmatrix}$$

so $\lambda_\Sigma(C) = |\mathrm{tr}(M(C))|$

Lemma (smoothing, ver. 1) If a multi-curve C_0 has a crossing with smoothings C_+ and C_-, then

$$\lambda(C_0) = \pm\lambda(C_+) \pm \lambda(C_-).$$

$\boxed{\text{usually } +}$



Pf (partial) Several cases: C_0 could be

consider last case.

Lemma For $A, B \in SL(2,\mathbb{R})$,

$$\mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}).$$

Pf A satisfies its characteristic polynomial

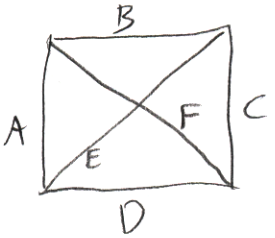
$x^2 - \mathrm{tr}(A)x + \underbrace{\det A}_{1} = 0$ $\quad * \quad A^2 - \mathrm{tr}(A)\cdot A + I = 0$

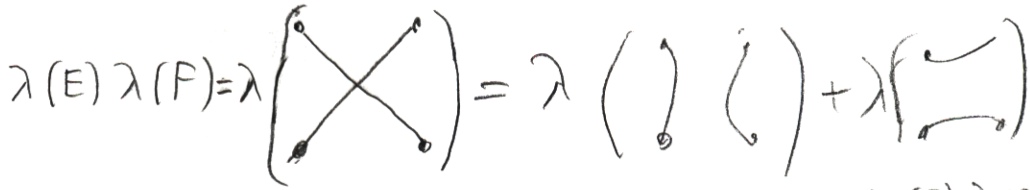
Multiply by $A^{-1}B$ and take trace.

$AB - \mathrm{tr}(A)B + A^{-1}B = 0$

$\mathrm{tr}(AB) - \mathrm{tr}(A)\,\mathrm{tr}(B) + \mathrm{tr}(A^{-1}B) = 0$

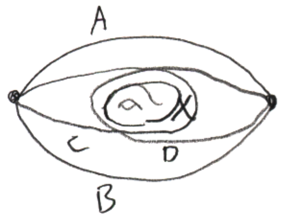
Example 1 Exchange:

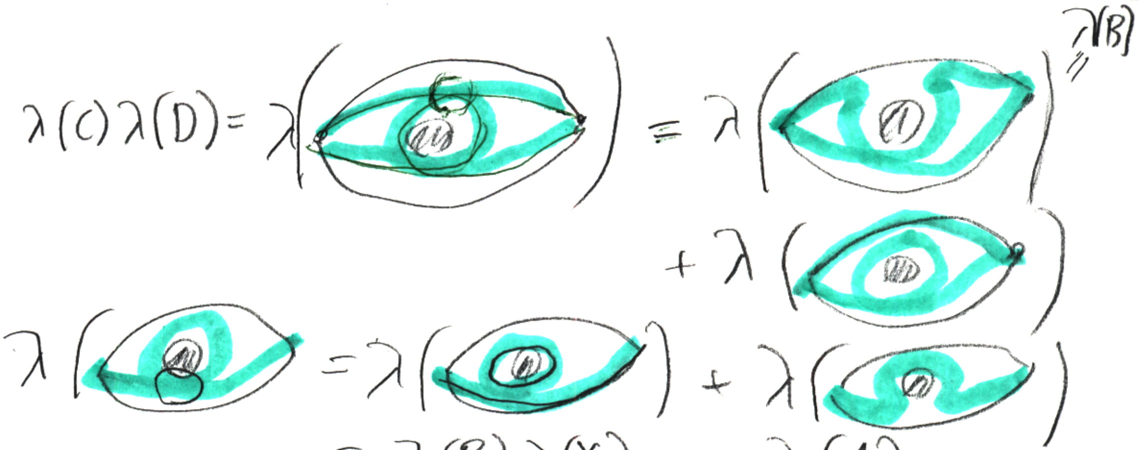




$$= \lambda(A)\lambda(C) + \lambda(B)\lambda(D)$$

Example 2

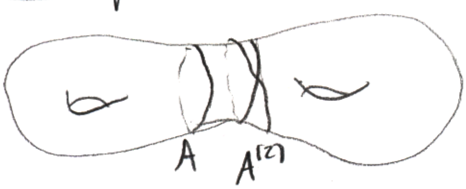


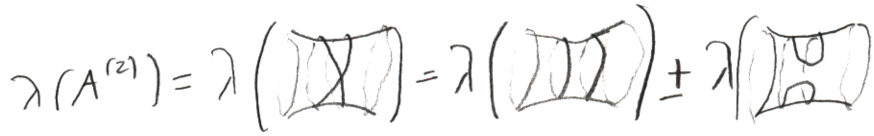


$$= \lambda(B)\lambda(X) \quad + \lambda(A)$$

$$\Rightarrow \boxed{\lambda(C)\lambda(D) = \lambda(B)^2 + \lambda(A)\lambda(B)\lambda(X) + \lambda(A)^2}$$

Example 3





$$= \lambda(A)^2 - 2$$

Why -2? $\quad M(A) \cong \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$

$$\lambda(A^{(2)}) = \left| tr\left(M(A)^2\right)\right| = tr\begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^{-2} \end{pmatrix} = \rho^2 + \rho^{-2} = (\rho + \rho^{-1})^2 - 2$$

$$= \lambda(A)^2 - 2.$$

In general: $\lambda(A^{(k)}) = 2T_k\left(\frac{\lambda(A)}{2}\right)$

$\uparrow$ Chebyshev polynomial of first kind

Prop For any multi-curve C,

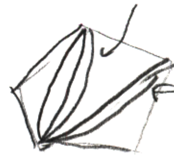
$$\lambda(C) = \sum_i \lambda(C_i)$$

where each C_i is a simple multi-curve (no intersections).

Pf Repeatedly apply smoothing lemma.

Fact $\{\lambda(C_i) \mid C_i$ a simple multi-curve$\}$ are linearly independent, (if no punctures)

E.g., cluster monomials are $\lambda(C_i)$:



$\rightsquigarrow \lambda(A)^3 \lambda(B)^2$

$\therefore \lambda(C_i)$ basis for some ring

Notions of positivity:

1) Positive Laurent phenomenon:
   - Each cluster variable is a Laurent polynomial in any other cluster.

   Conj This Laurent polynomial has positive coefficients (for any cluster algebra)

   Thm (Musiker–Schiffler) This is true for cluster algs. from surfaces w/o punctures

2) Positive basis: Linear basis $\{x_i\}$ containing cluster monomials so $\forall i, j$,

$$x_i x_j = \sum a_{ij}^k x_k$$

   where $a_{ij}^k \geq 0$.

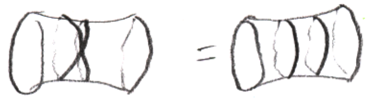
3) Canonical basis: Basis in (2) agree with another basis (e.g., Lusztig)

4) Upper cluster algebra: Basis for elements which are positive Laurent polys. with respect to each cluster.

All different!

Is basis from simple multi-curves a positive basis?

No: $\lambda(A^{(2)}) = \lambda(A)^2 - 2$

 $- 2$

<u>Def</u> The <u>signed λ-length</u> $\lambda^{\pm}(C)$ of an immersed multi-curve C obeys:

- $\lambda^{\pm}(\bigcirc) = -2$ ← contractible loop
- $\lambda^{\pm}(\rho) = 0$
- $\lambda^{\pm}(\ell) = -\lambda^{\pm}(\frown)$
- $\lambda^{\pm}(C) = \lambda(C)$ if C is <u>taut</u>: none of features above

<u>Lemma</u> $\lambda^{\pm}(\times) = \lambda^{\pm}(\,)(\,) + \lambda^{\pm}(\asymp)$

<u>Def</u> A multi-curve is <u>compatible</u> (with itself) if

- No two components intersect
- Each component is simple or a power of a simple circle
- No two components are powers of the same simple circle $A^{(k)} \cup A^{(\ell)}$ not allowed

<u>Conj</u> $\{\lambda(C) \mid C$ a compatible multi-curve$\}$ is a positive basis.

<u>Note</u> For $(S, M) =$  (Quiver ),

$\langle \{\lambda(C) \mid C$ compatible multi-curve$\} \rangle \neq$ ring gen. by cluster variables

$\subsetneq$ upper cluster algebra

finitely gen as rings (allowing signs)

Not finitely gen. as algebra

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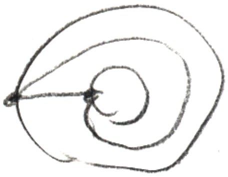
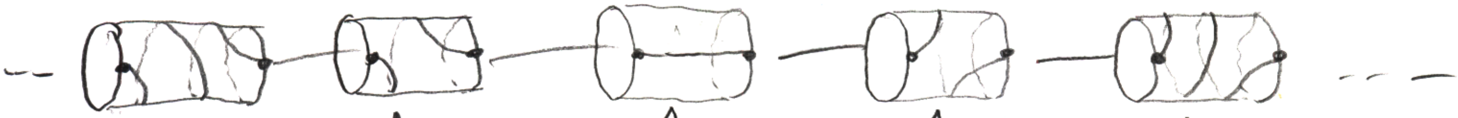
Example For $(S,M) =$ , $\mathcal{A}(S,M) = \tilde{A}_{1,1}$ 

compatible multi-curves:



A_{-2} A_{-1} A_0 A_1 A_2

and (k copies of A_i) $\cup$ (l copies of A_{i+1})

C $C^{(2)}$ $C^{(3)}$

Set $y_i = \lambda(A_i)$

$z_i = \lambda(C^{(i)})$ $\qquad z_i = 2T_i(\frac{z_1}{2})$

Recovers cluster algebra studied completely by Sherman-Zelevinsky.

Exercise Check Sherman-Zelevinsky relations:

$y_{i-1}y_{i+1} = y_i^2 + 1$ (setting coefficients to 1, i.e.,

$y_{i-1}y_{i+2} = y_i y_{i+1} + z_1$ λ(boundary) = 1)

Thm (Sherman-Zelevinsky) The y_i, z_i are a basis for ~~upper cluster algebra~~. positive cone

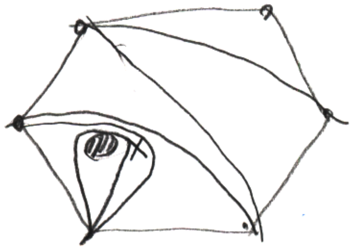
But: Lusztig's dual canonical basis for $\tilde{A}_{1,1}$ has Chebyshev polynomials of second kind, not first.

E.g., $\tilde{z}_2 = z_1^2 - 1$ vs $z_2 = z_1^2 - 2$

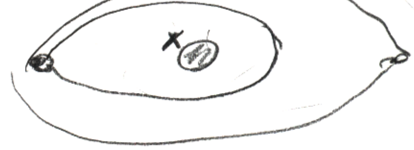
(Le Clerc)

II. Non-simply-laced cluster algebras from surfaces

Suppose we allow boundaries with no punctures.

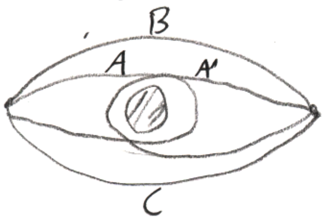


Boundary always included in unique monogon in any maximal collection of arcs.

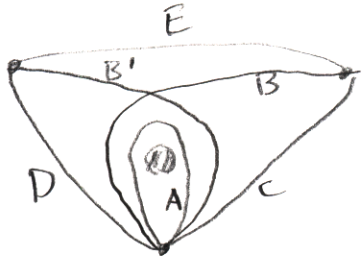


$\lambda(X)$ is fixed parameter.

Exchange relations:



$$\lambda(A)\lambda(A') = \lambda(B)^2 + \lambda(x)\lambda(B)\lambda(C) + \lambda(C)^2$$



$$\lambda(B)\lambda(B') = \lambda(A)\lambda(E) + \lambda(C)\lambda(D)$$

Not a cluster algebra!

Two special cases:

- $\lambda(X) = 0$: $\lambda(A)\lambda(A') = \lambda(B)^2 + \lambda(C)^2$

Above case: B-matrix

$$\begin{array}{cl} \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 2 \\ & & & & -1 & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow B \\ \leftarrow A \end{array} \end{array}$$

$\boxed{C_n}$

- $\lambda(X) = 2$: $\lambda(A)\lambda(A') = \lambda(B)^2 + 2\lambda(B)\lambda(C) + \lambda(C)^2 = (\lambda(B)+\lambda(C))^2$

New "arc" $A^{1/2}$ so $\lambda(A^{1/2}) := \sqrt{\lambda(A)}$:

$$\lambda(A^{1/2})\lambda((A')^{1/2}) = \lambda(B) + \lambda(C).$$

$\boxed{B_n}$

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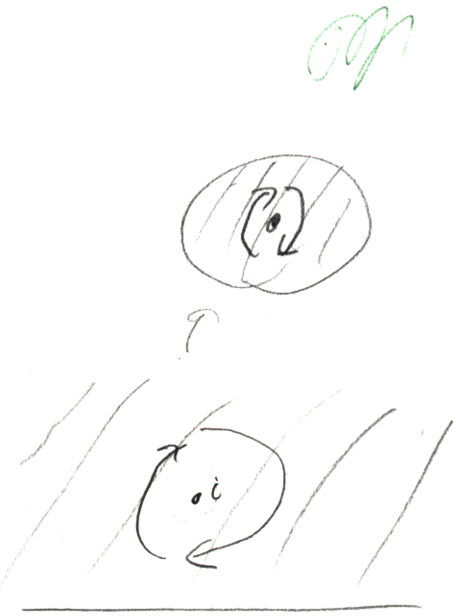
Above cases:
B-matrix $\begin{pmatrix} \ddots & & & & \\ & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -2 & 0 \end{pmatrix}$



Geometric interpretation

- $\lambda(X) = 0$: want $tr(M(X)) = 0$

$$M(X) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on $\mathbb{H}^2$, acts by $z \mapsto -\frac{1}{z}$

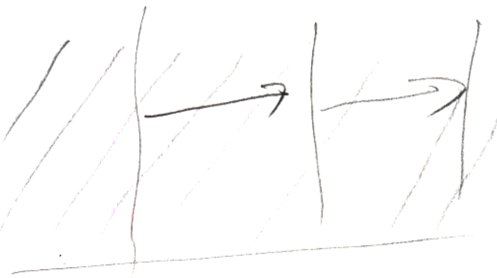
rotation by $180°$

Quotient plane $\mathbb{H}^2$ by rotation to get cone point of order 2.

- $\lambda(X) = 2$: $|tr(M(X))| = 2$

$$M(X) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$$z \mapsto z + c$$



X is a puncture with cusp, but treated differently.