Constructing cluster algebra from surfaces

Today:

Construct cluster algebras from surfaces which are

- Independent of coefficients
- Concrete description
- Finite mutation type: only finitely many exchange matrices

Can classify cluster complex:

- Polynomial or exponential growth (usually exponential)
- Contractible or not (usually contractible, otherwise $\simeq$ a sphere)

These give all but finitely many $\underbrace{\text{cluster}}_{\text{skew-symmetric}}$ algebras of finite mutation type.

Constructing the Cluster algebra for surfaces

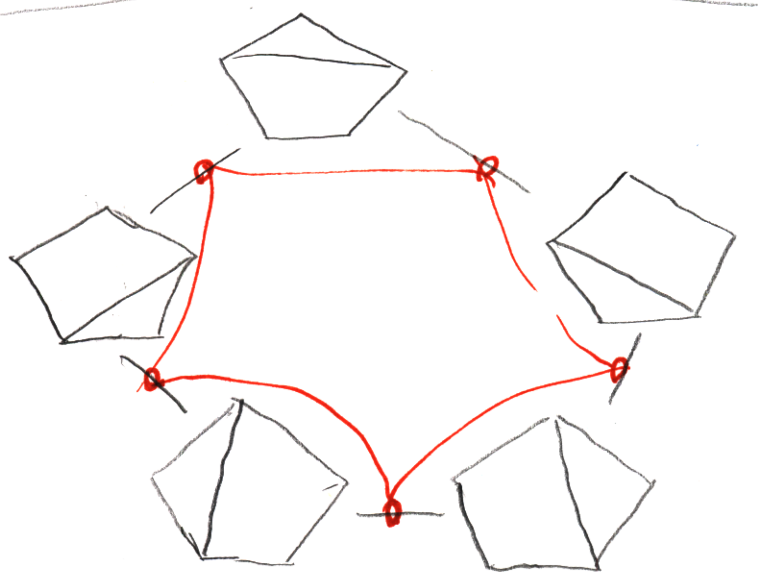
Recall that arcs on a surface S with marks $M \subset S$ are embedded intervals with endpoints at M.
Two arcs are compatible if they do not intersect.

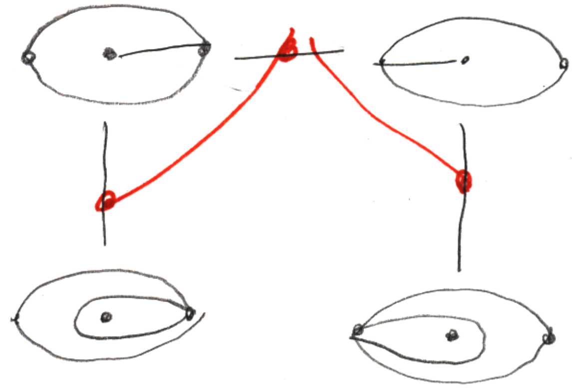
Def The arc complex $\Delta(S,M)$ of (S,M) is the complex with

- Vertices: arcs in (S,M)
- Edges connect pairs of compatible arcs
- Higher faces from cliques: sets of compatible arcs

In particular, highest faces are triangulations.
This is a simplicial complex.

Examples

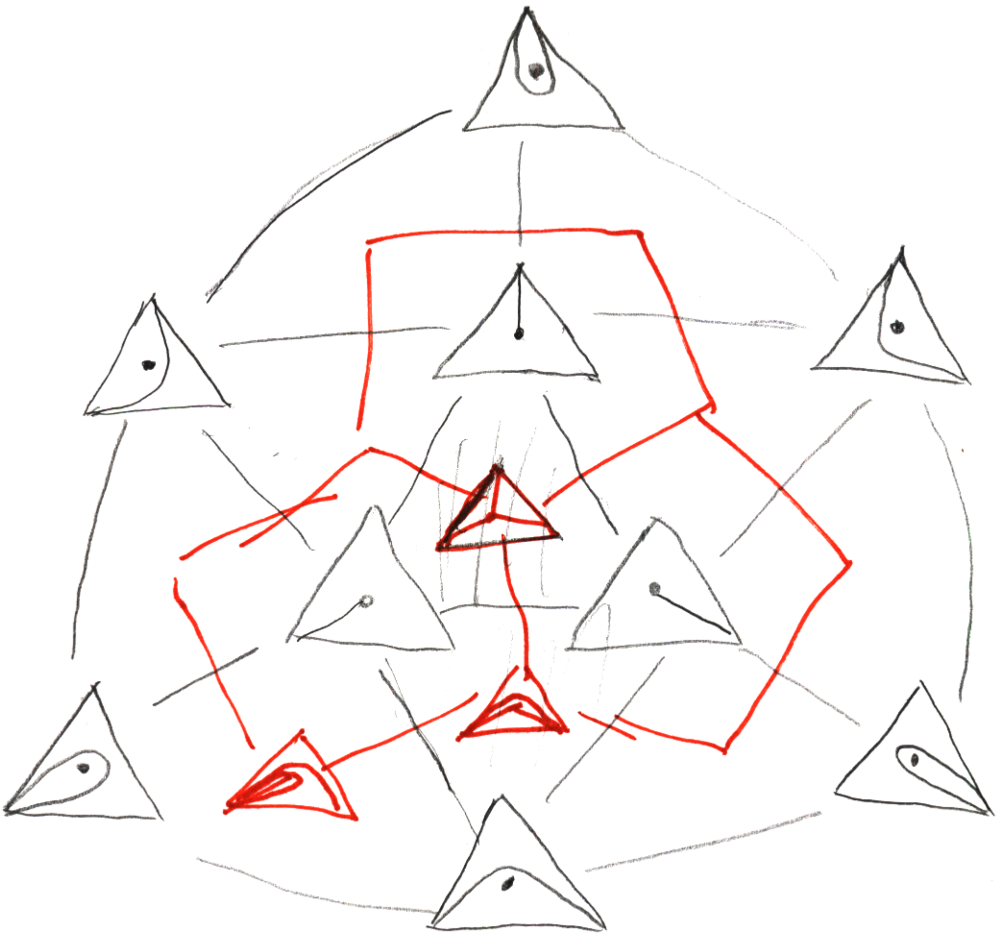




Def The triangulation graph $T(S,M)$ is the dual graph to the arc complex:

- Vertices: Triangulations
- Edges connect vertices related by an edge flip

(Can also define higher faces via associahedra)



Topological facts

Prop The arc complex $\Delta(S,M)$ is a pseudo-manifold (with boundary):
Each $(n-1)$-simplex is contained in (1 or) 2 n-simplices,
where n = maximal dim. of any simplex

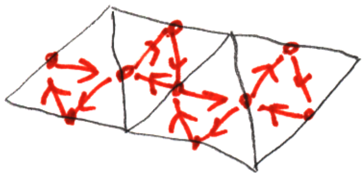
Thm (Harer, Hatcher) $\Delta(S,M)$ is contractible unless
(S,M) is a polygon (i.e., S = disk, $M \subset \partial S$)

How to complete so that $\Delta(S,M)$ has no boundary
(so $T(S,M)$ is n-regular)?

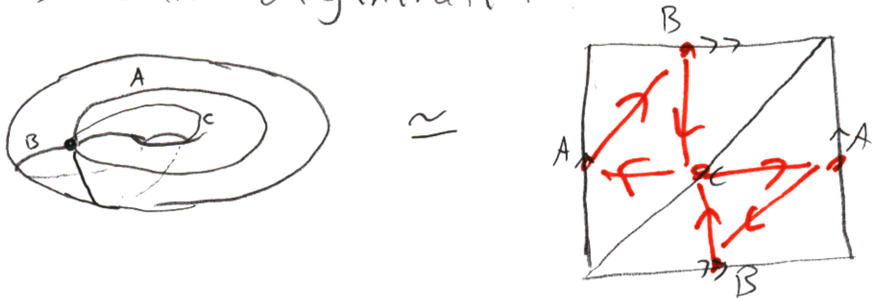

Stuck!

Construction of B matrix (1st version)

From triangulation:
- one node per edge
- edges clockwise in each triangle

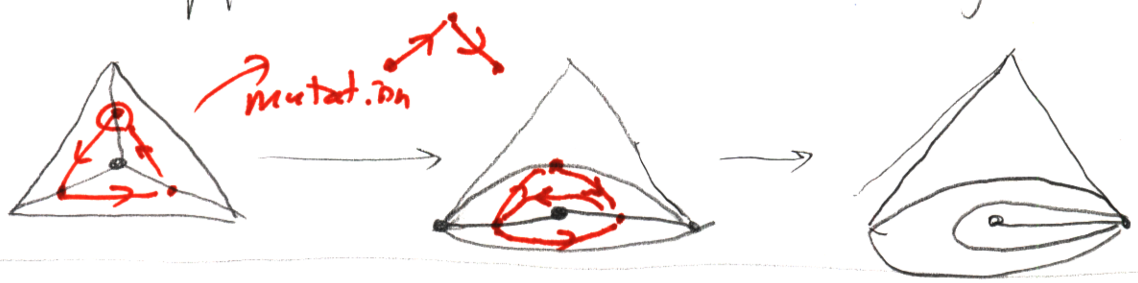


This can degenerate:



$$\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

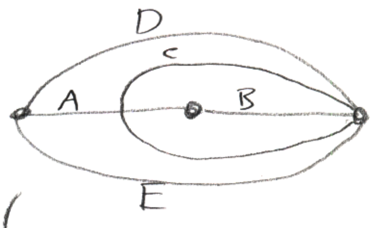
What happens for self-folded triangles?



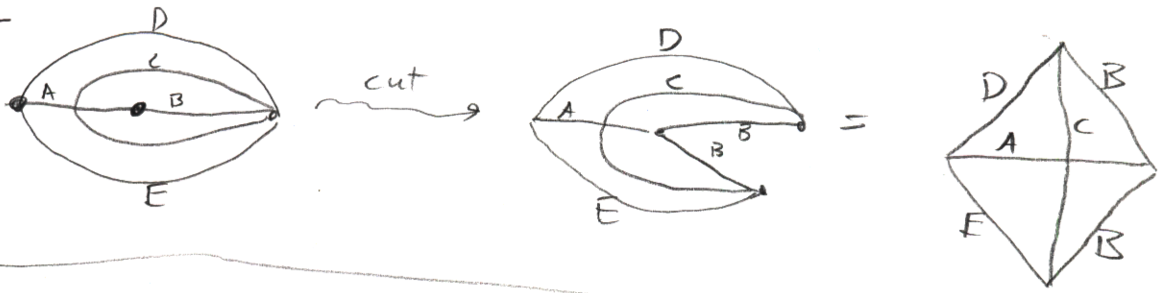
Lemma Inside a bigon, we have

$$\lambda(A)\frac{\lambda(C)}{\lambda(B)} = \cancel{\lambda(B)}\lambda(D) + \cancel{\lambda(B)}\lambda(E).$$

Not relatively prime!



Pf



Question What is $\frac{\lambda(C)}{\lambda(B)}$?

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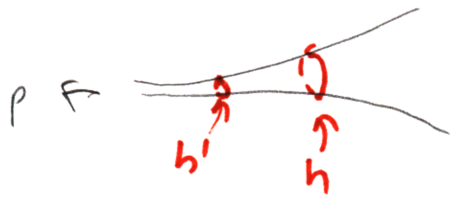
Back to geometry...

Def For $\Sigma \in \mathcal{T}(S,M)$ a geometric structure on (S,M),

$p \in M$ a puncture,

h a horocycle around p,

let $L(h)$ be the horocyclic length: the length of h (as a curve)

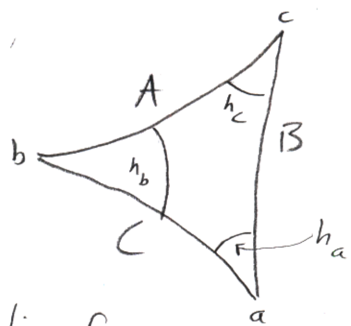


Def Two horocycles h, h' around p are conjugate if $L(h)\,L(h')=1$.

(Remember curvature is -1, so lengths have absolute meaning.)

Lemma In a decorated triangle,

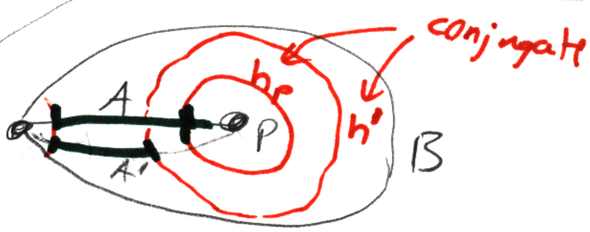
$$L(h_a) = \left(\frac{\lambda(A)}{\lambda(B)\lambda(C)}\right)$$



Pf Look at scaling behaviour as horocycles are changed: unchanged for b, c, correct scaling for a. Do an integral to fix constant.

Cor In a monogon,

$$L(h_p) = \left(\frac{\lambda(A)^2}{\lambda(B)}\right)^{-1}$$



Cor $\lambda(B) = \lambda(A)\,\lambda(A')$, where $\lambda(A')$ is lambda-length to conjugate horocycle at p.

Pf $L(h_p)^{-1} = \frac{\lambda(A)^2}{\lambda(B)}$

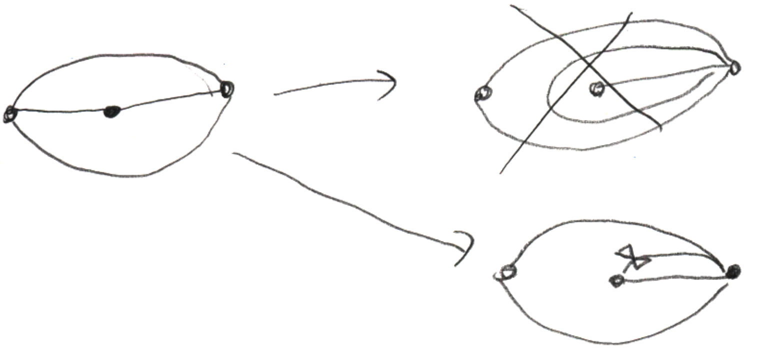
$$L(h_p)^{-1} L(h_p')^{-1} = 1 = \frac{\lambda(A)^2\lambda(A')^2}{\lambda(B)^2}.$$

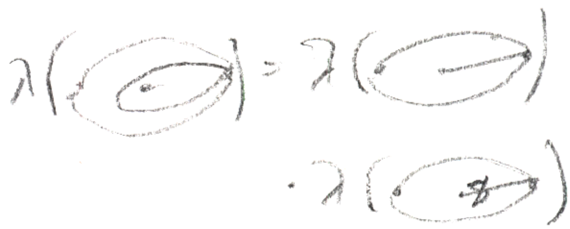
$L(h_p')^{-1} = \frac{\lambda(A')^2}{\lambda(B)}$

<u>Def</u> A <u>tagged arc</u> is an arc A with one or both ends $^{\text{may be}}$ decorated by <u>tags</u> $\bowtie$ so that:

- A does not enclose a punctured monogon
- If both endpoints of A at same mark, both tagged or both untagged.
- Endpoints at boundary marks are untagged.

<u>Think:</u> arc to conjugate horocycle.

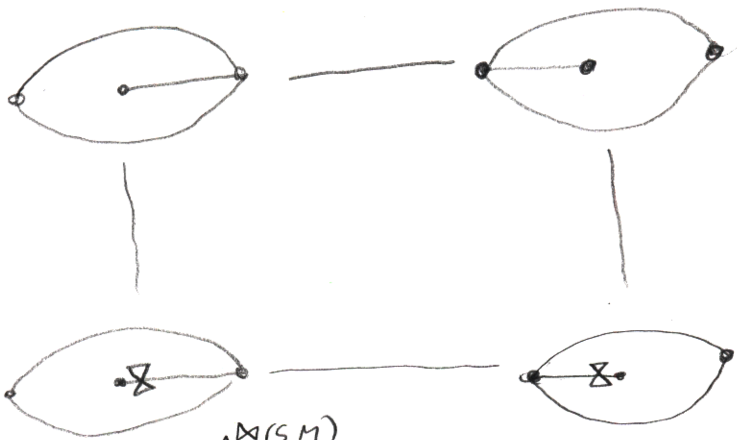




<u>Def</u> Two tagged arcs A, B are <u>compatible</u> if

- Untagged versions are compatible
- If A, B share an endpoint p, tag of A at p = tag of B at p

<u>unless</u> untagged arcs and other endpoint agree.

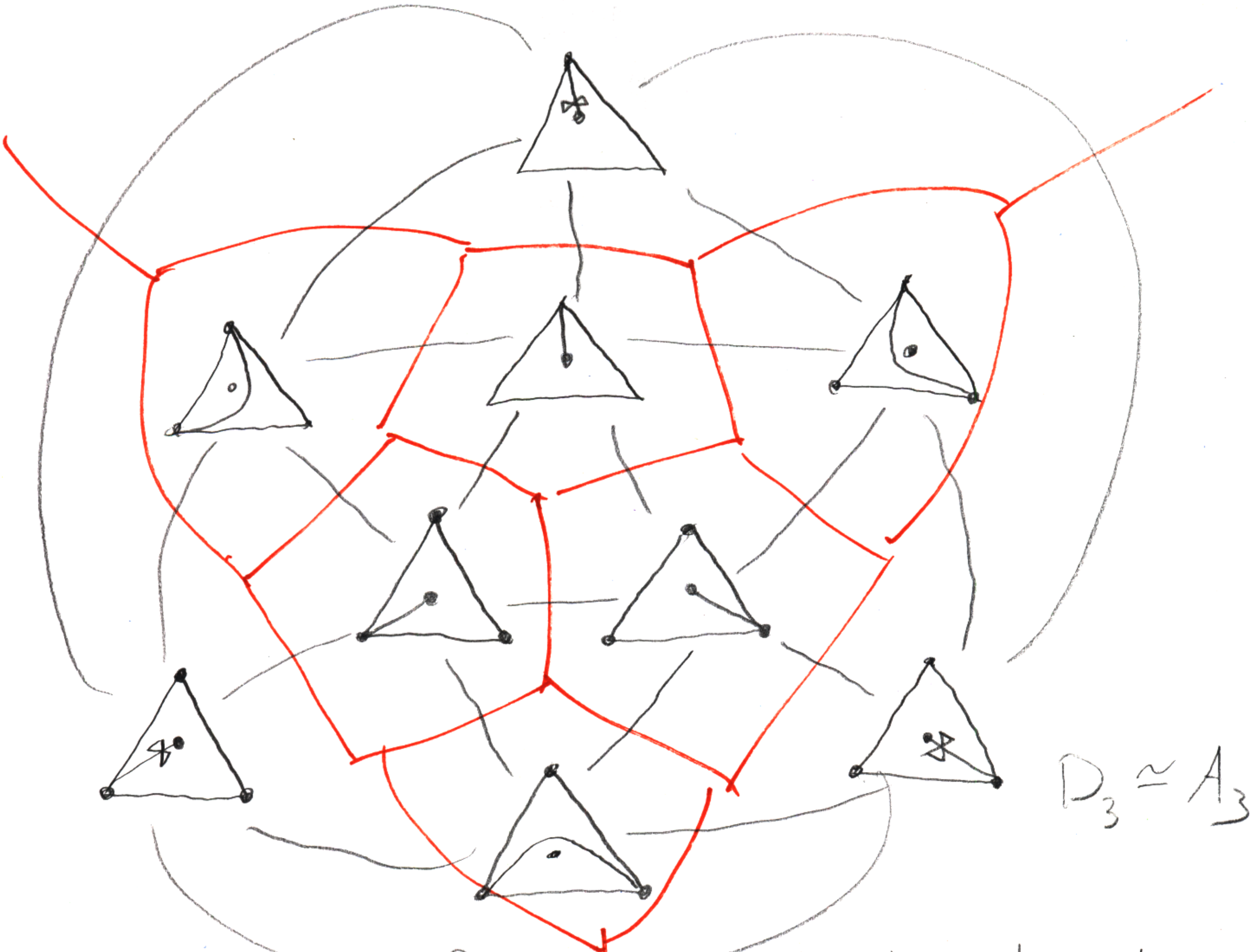


<u>Def</u> Tagged arc complex $\Delta^{\bowtie}(S,M)$, tagged triangulation graph $T^{\bowtie}(S,M)$ as before.

<u>Prop</u> $\Delta^{\bowtie}(S,M)$ is a pseudomanifold.

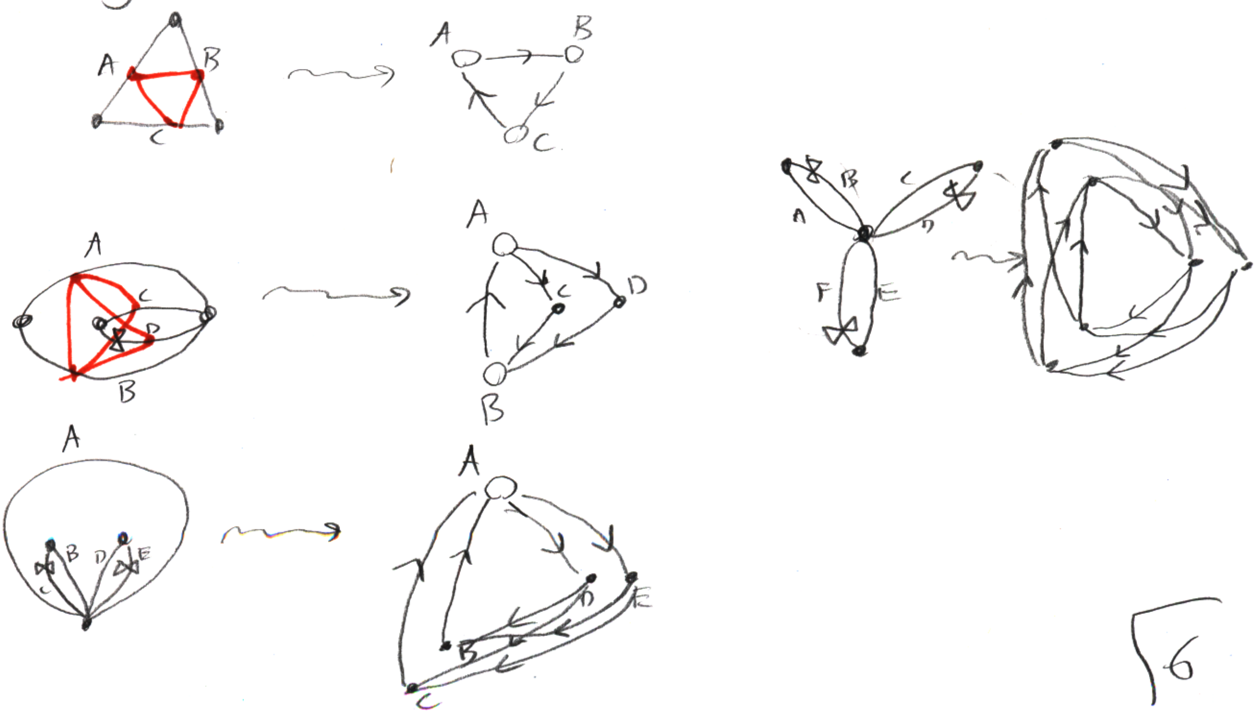
(each $n-1$ simplex is in 2 n-simplices)

$\Rightarrow T^{\bowtie}(S,M)$ is regular



$D_3 \simeq A_3$

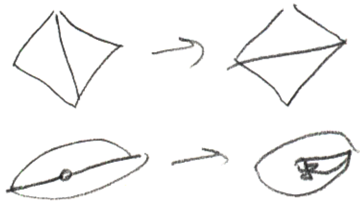
Def The $\underline{\text{B-matrix } \mathcal{B}(T)}$ associated to a tagged triangulation T of (S,M) is obtained by gluing following blocks at open vertices and removing boundary vertices!



Thm (Fomin–Shapiro–T) If S is not a closed surface with two punctures, there are cluster algebras $\mathcal{A}(S,M)$ with

Later removed

- exchange graph $= T^{\bowtie}(S,M)$
  - seeds $\rightsquigarrow$ tagged triangulation
  - mutation $\rightsquigarrow$ flip (edge or bigon)
  - B-matrix $= B(T)$



- cluster complex $= \Delta^{\bowtie}(S,M)$

(arbitrary coefficients)

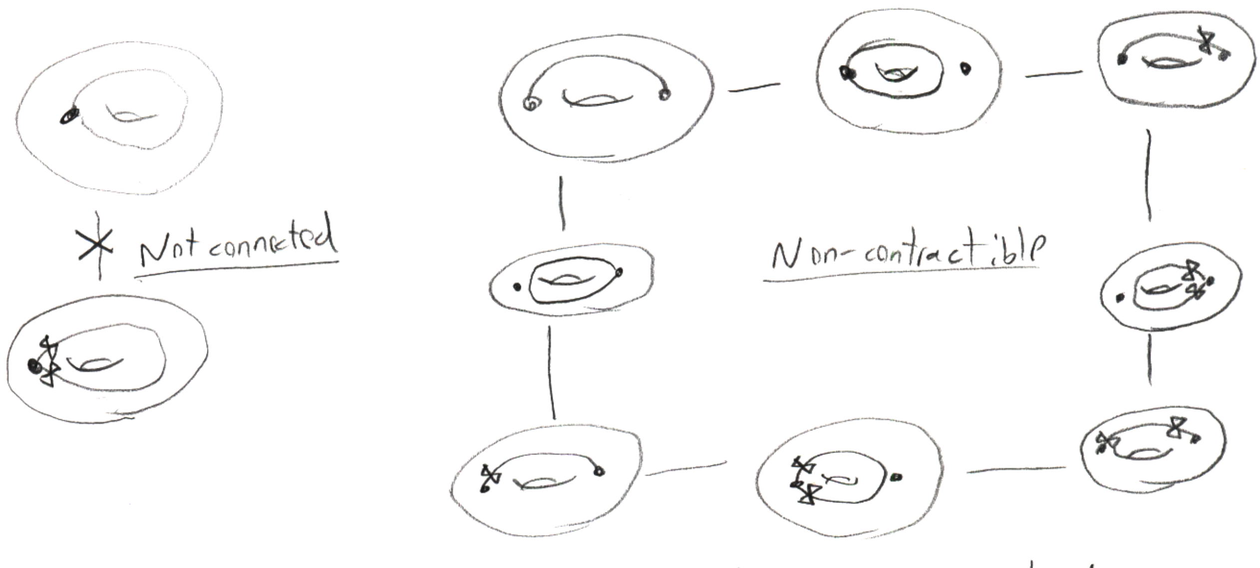
Proof looks at connectivity: (independent of geometry):

Thm Arc complex $\Delta^{\bowtie}(S,M)$ is contractible unless S is a closed surface, in which case $\Delta^{\bowtie}(S,M) \simeq S^{|M|-1}$

(or (S,M) finite type)

sphere of dimension $|M|-1$

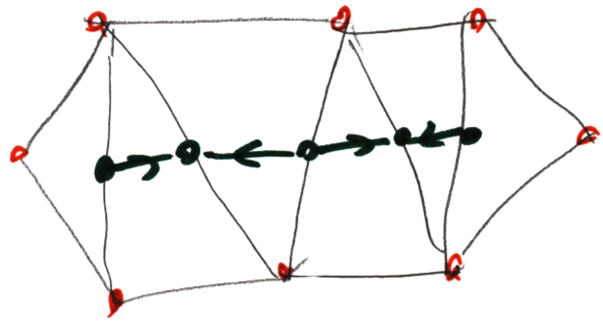
In particular, $\Delta^{\bowtie}(S,M)$ is $\left\{\begin{matrix}\text{connected} \\ \text{simply connected}\end{matrix}\right\}$ unless S closed, $|M| = \left\{\begin{matrix}1 \\ 2\end{matrix}\right\}$



In good cases, $T^{\bowtie}(S,M)$ is also contractible using only loops from $A_1 \times A_1$ or A_2, which we understand.

$\underline{A_n}$

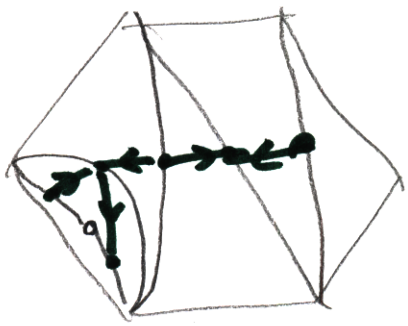
polygon w/ n+3 sides



~~A_3~~ A_5

$\underline{D_n}$

Punctured polygon w/ n sides



D_6

arXiv:0811.1703

Thm (Felikson-Shapiro-Tumarkin) All but finitely many cluster algebras with a skew-symmetric B matrix of finite mutation type are

- of rank 2, or
- come from the surface construction.

Exceptions

Dynkin diagrams

E_6: 

E_7: 

E_8: 

Affine Dynkin diagrams

$\tilde{E}_6$: 

$\tilde{E}_7$: 

$\tilde{E}_8$: 

Extended Affine Dynkin diagrams

$E_6^{(1,1)}$ 

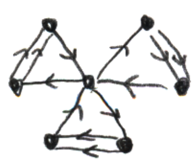
$E_7^{(1,1)}$ 

$E_8^{(1,1)}$ 

(Triangles are oriented)

Weirdos

X_6: 

X_7: 

(found by Owen-Derksen)

<u>Thm</u> If $\mathcal{A}(S,M)$ is of polynomial growth, it is

- finite type (A_n, D_n) polygon, punctured polygon
- linear growth $(\tilde{A}_{n,m}, \tilde{D}_n)$ annulus, ~~punctured annulus~~ twice punctured polygon
- quadratic growth
- cubic growth

listed explicitly

Otherwise of exponential growth

Proof looks at <u>mapping class group</u>: maps $(S,M) \to (S,M)$ modulo isotopy fixing M.

Well-studied, exponential growth unless (S,M) small.