Hyperbolic Geometry and λ-coordinates

Today: Geometric cluster algebra from hyperbolic structures on surfaces

Measured laminations are the tropical limit

Construction due to Fock - Goncharov

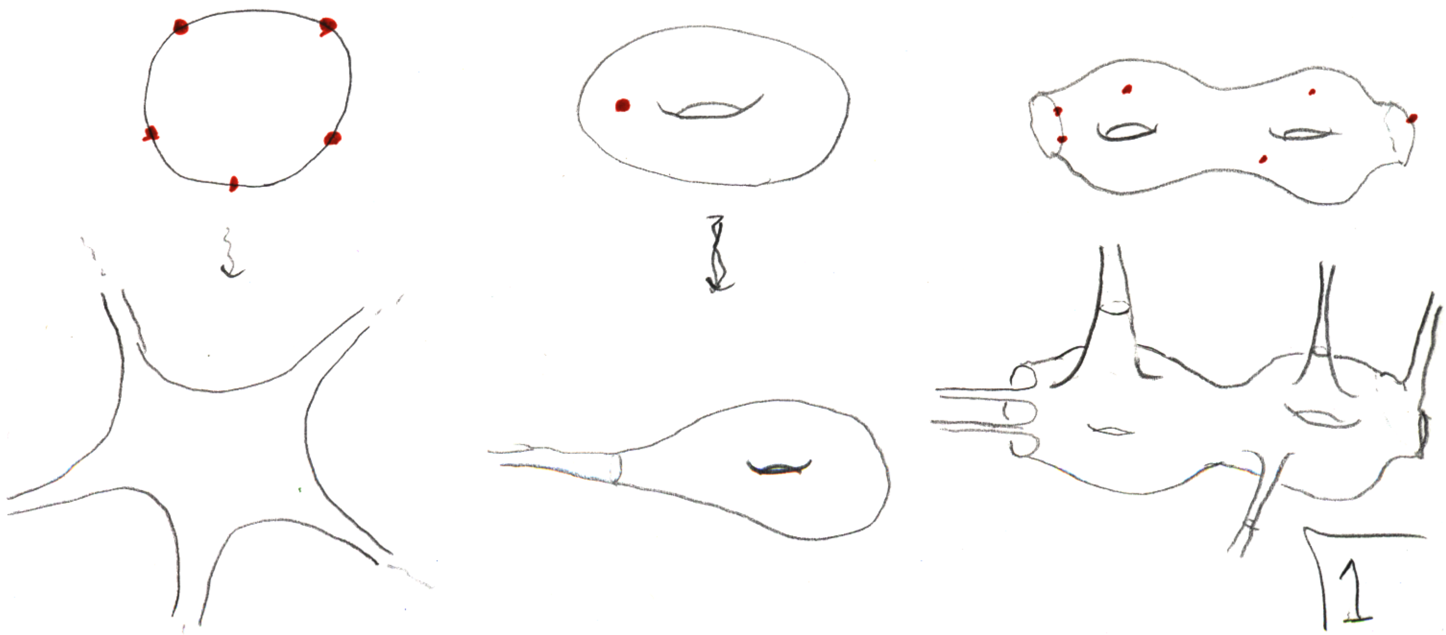
Gekhtman - Shapiro - Vainshtein

Def Given (S, M) marked surface as before, not too small.

The Teichmüller space $\mathcal{T}(S, M)$ is the space of metrics on $S - M$ which

- are hyperbolic: constant curvature -1
- have geodesic boundary at boundary of S
- have cusps at points in M: go off to infinity ($S - M$ is complete) while area remains bounded

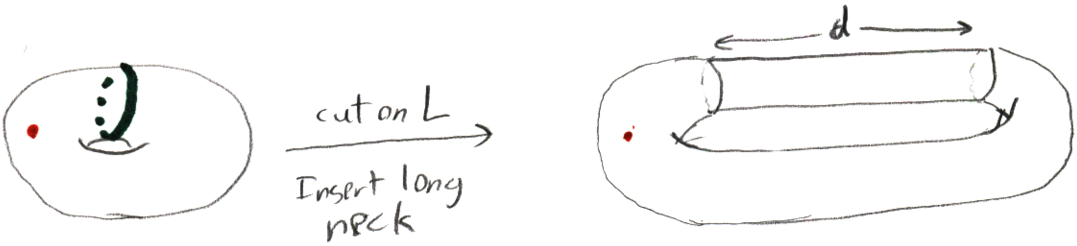
considered up to diffeomorphisms homotopic to identity. ($\mathrm{Diff}_0(S, M)$)

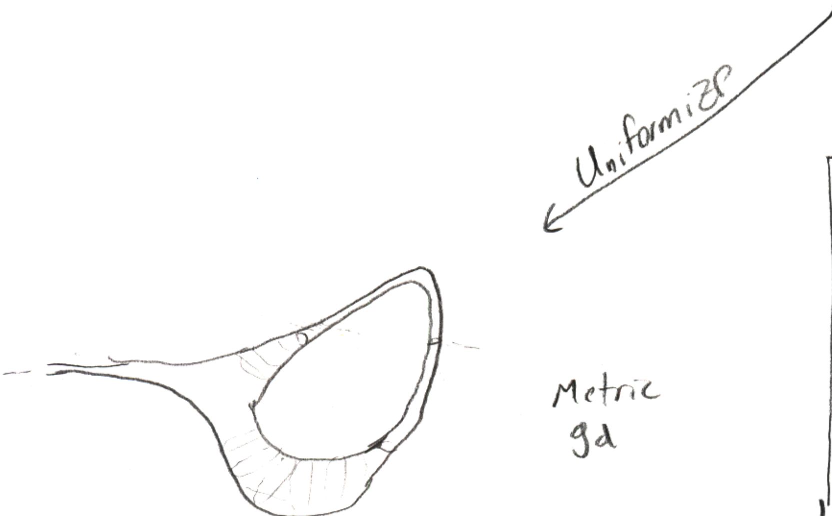


Notes

1. Equivalent to conformal structure on S by uniformization theorem.

2. Turn a measured lamination L into a family of metrics:





As $d \to \infty$, lengths in g_d are dominated by # of times curve crosses L.

Projective measured laminations form the (W.) <u>Thurston boundary</u> of Teichmüller space.

3. Contrast Teichmüller space with <u>moduli space</u>: quotient by <u>all</u> diffeomorphisms, metrics up to equivalence.
   - Teichmüller space records how metric sits on surface
   - Teichmüller space is contractible, moduli space is not
   - Moduli space has finite volume, Teichmüller space does not.

4. $\mathcal{T}(S,M)$ is a manifold of dimension

$$\underset{\text{genus}}{6g} - 6 + 2\underset{\text{punctures}}{p} + 3\underset{\text{boundary components}}{b} + \underset{\text{marked pts on boundary}}{c}$$

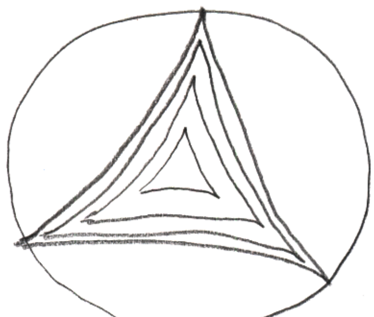
Example Ideal polygons

$S = \text{disk}, \quad M \subset \partial S$

$|M| = c$

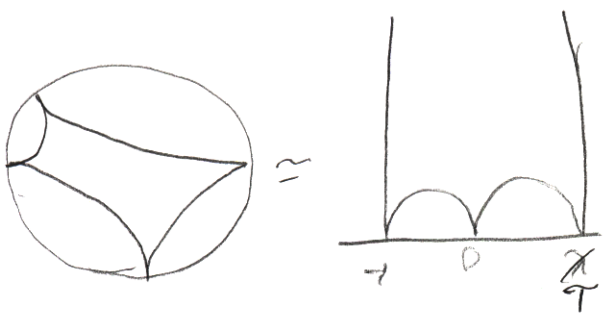
Think of hyperbolic structure as subset of hyperbolic plane.

c=3: Get ideal triangle



- Limit of finite triangles
- Area $= \pi$
- Only one up to isometry ($PSL(2,\mathbb{R})$ acts simply transitively on triples of ideal pts.)

c=4: ideal quadrilateral



Put three corners at $0, -1, \infty$. Remaining corner is at a point $T \in \mathbb{R}_{>0}$: invariant of quadrilateral, the cross-ratio

⇝ shear coordinates

Later.

Models for hyperbolic plane $\mathbb{H}^2$

Upper half-plane

Metric

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

Geodesics: circles perpendicular to boundary.

Symmetries: $PSL(2,\mathbb{R})$, acting by

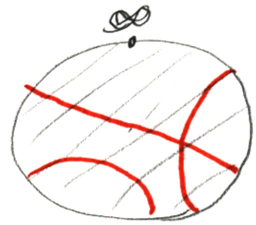
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

(fractional linear transformation)

Poincaré disk

Metric

$$ds^2 = \frac{dx^2+dy^2}{(1-r^2)^2}$$



Geodesics: circles perpendicular to boundary.

Related to upper half-plane model by (complex) fractional linear transform.

Easier to draw.

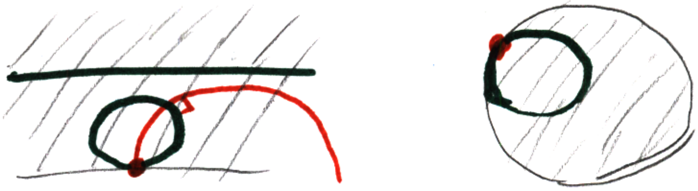
To get numbers associated to arcs, need to <u>renormalize</u>

<u>Def</u> A <u>horocycle</u> (at ideal point p) is set of points "equidistant to p"

In lift to $\mathbb{H}^2$, looks like circle tangent to boundary at p

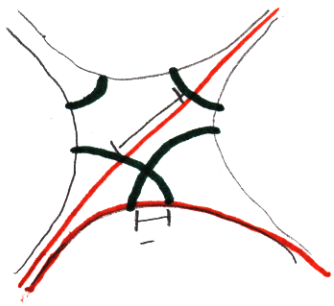
(limit of finite circles)

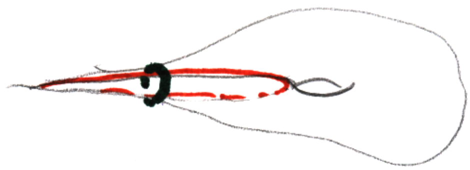
Perpendicular to every geodesic to p



<u>Def</u> The <u>decorated Teichmüller space</u> $\widetilde{\mathcal{T}}(S,M)$ is

- a point in $\mathcal{T}(S,M)$
- a choice of horocycle around each cusp from M





<u>Def</u> (Penner) For an arc A on (S,M) and $\Sigma \in \widetilde{\mathcal{T}}(S,M)$, the <u>length of A with respect to Σ</u> is

$$\underset{\mathbb{R}}{\underset{\cap}{\ell_\Sigma(A)}} = \text{length on geodesic representative of } A \text{ between intersections with horocycles around ends}$$

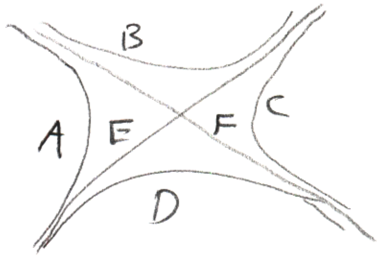
(negative if horocycles intersect)

The <u>λ-length</u> is

$$\underset{\mathbb{R}_{>0}}{\underset{\cap}{\lambda_\Sigma(A)}} = \exp(\ell_\Sigma(A)/2)$$

Lemma In ideal quadrilateral,

$$\lambda_\Sigma(E)\,\lambda_\Sigma(F)=\lambda_\Sigma(A)\,\lambda_\Sigma(C)+\lambda_\Sigma(B)\,\lambda_\Sigma(D).$$



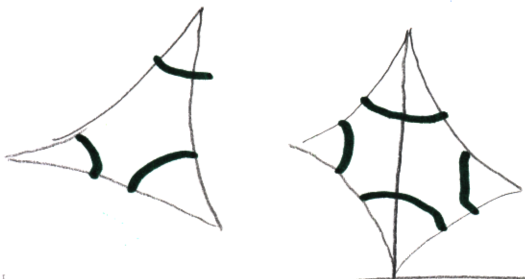
Theorem (Penner) For any triangulation $(A_i)_{i=1}^{t}$ of (S,M) with boundary arcs $(B_i)_{i=1}^{c}$,

$$\Sigma \longmapsto (\lambda_\Sigma(A_i))_{i=1}^{t} \sqcup (\lambda_\Sigma(B_i))_{i=1}^{c}$$

$$\widetilde{\mathcal{T}}(S,M) \longrightarrow \mathbb{R}_{>0}^{t+c}$$

is a homeomorphism.

Proof sketch Decorated triangle is unique given lengths. Unique way to glue two adjacent triangles.



Therefore get (partial) realization of cluster algebra inside functions on $\widetilde{\mathcal{T}}(S,M)$

base field $\rightsquigarrow$ suitable functions on $\widetilde{\mathcal{T}}(S,M)$

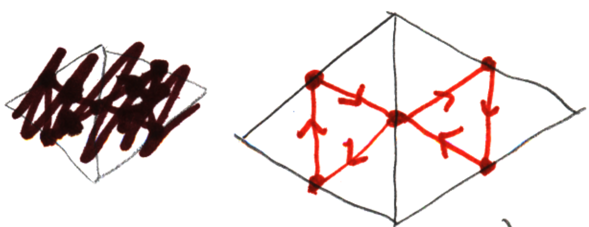
seeds $\rightsquigarrow$ triangulations

cluster variables $\rightsquigarrow$ λ-lengths $\lambda_\Sigma(A)$ (as function on $\widetilde{\mathcal{T}}(S,M)$)

mutation $\rightsquigarrow$ edge flip

coefficients $\rightsquigarrow$ boundary arcs

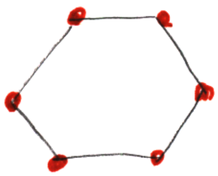
B-matrix $\rightsquigarrow$



(more or less)

All surfaces have only finitely many combinatorial types of triangulations ($\rightsquigarrow$ cluster algebra mutationally finite)

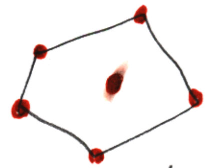
A few surfaces have only finitely many triangulations



Disk w/ n boundary points

$\updownarrow$

A_{n-3}



Disk w/ one puncture n boundary points

$\updownarrow$

D_n

Example: A_n: Usual realization:

$$Gr(2,n+3) = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n+3} \\ b_1 & b_2 & \cdots & b_{n+3} \end{bmatrix} \Big/ SL(2)$$

w/ Plucker coordinates.

Ratio $\frac{a_i}{b_i} \in \mathbb{R}$, on boundary of $\mathbb{H}^2$ (upper half-plane model)

$(n+3 \text{ points in } \mathbb{R}) / SL(2,\mathbb{R}) \simeq$ space of ideal $(n+3)$-gons

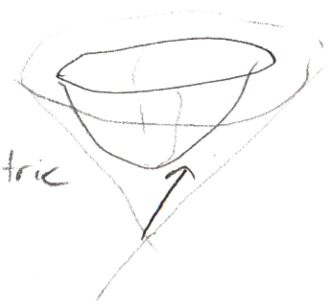
Pair (a_i, b_i) also has a scale $\rightsquigarrow$ horocycle

determinant $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ is λ-length.

Hyperboloid model

$x^2+y^2-z^2=-1 \qquad z>0$

w/ induced Lorentzian metric



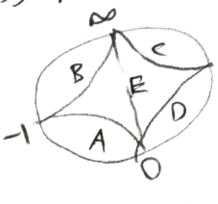
$PSL(2,\mathbb{R}) \simeq SO^+(2,1)$

Shear coordinates

Suppose we forget horocycles and work with original Teichmüller space.

Def Given a hyperbolic structure $\Sigma \in \mathcal{T}(S,M)$ and triangulation $T=(A_i)_{i=1}^{t}$, the shear coordinates are

$$\tau_\Sigma(E;T) = \text{cross-ratio of quadrilateral}$$



$\leftarrow \tau_\Sigma(A_i;T)$

$$= \frac{\lambda_{\tilde\Sigma}(A)\,\lambda_{\tilde\Sigma}(C)}{\lambda_{\tilde\Sigma}(B)\,\lambda_{\tilde\Sigma}(D)} \quad \text{for any } \tilde\Sigma \in \tilde{\mathcal{T}}(S,M) \text{ lifting } \Sigma.$$

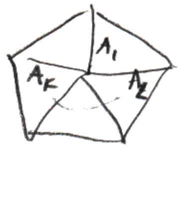
Thm The map

$$\Sigma \longmapsto \left(\tau_\Sigma(A_i;T)\right)_{i=1}^{t}$$

$$\mathcal{T}(S,M) \longrightarrow \mathbb{R}^t$$

is a homeomorphism onto subset where, for each puncture w/ incident arcs $A_1,\dots,A_k$

$$\prod_{i=1}^{k} \tau_\Sigma(A_i;T) = 1$$



Note: Can drop this and parametrize larger space with boundary at punctures