A SHADOW CALCULUS FOR 3-MANIFOLDS

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Abstract. We recall Turaev's theory of shadows of 4-manifolds, and its use to present 3-manifolds. We then prove a calculus for shadows of 3-manifolds which can be viewed as the analogous of Kirby calculus in the shadow world. This calculus has the pleasant feature of being generated only by local moves on the polyhedra.

Contents

1. Introduction

Shadows were defined by V. Turaev for the first time at the beginning of the nineties in [15] as a method for representing knots alternative to the standard one based on knot diagrams and Reidemeister moves. The theory was then developed in the preprint "Topology of shadows" ([13]) which was later included in a revisited version in [12]; moreover, a short account of the theory was published by Turaev in [14]. Since then, probably due to the slightly higher degree of complication of this theory with respect to Kirby calculus, only few applications of shadows were studied. Among these applications we recall the use of shadows to study Jones-Vassiliev invariants of knots made by U. Burri in [1] and A. Shumakovitch in [11] and the study of "Interdependent modifications of links and invariants of finite degree" developed by N.M. Goussarov in [6].

It is our conviction that the potentialities of shadows are still to be unravelled. The present paper is devoted to introduce the reader to shadows as a tool to study 3-manifolds and then to prove a calculus for these objects which represents the analogous in the shadow world of the Kirby calculus. The subsequent paper [3] will be devoted to define a new notion of complexity of 3-manifolds based on shadows which turns out to be intimately connected with hyperbolic geometry in dimension 3.

Roughly speaking, a shadow of a 4-manifold M is a spine of the manifold, that is a 2-dimensional polyhedron X embedded in the manifold so that M collapses on X. In dimension 3, a spine of a 3-manifold allows one to fully reconstruct the 3-manifold from the combinatorial structure of the polyhedron; it is not difficult to check that this is false in dimension 4. For instance, consider the particular polyhedron homeomorphic to S^2: one can embed it as a zero section both in $S^2 \times D^2$ and in $S^2 \times_1 D^2$ (the second space being the disc bundle over S^2 with Euler number 1, i.e. $\overline{\mathbb{CP}^2} - B^4$). In both cases the embedded S^2 is a spine of the two manifolds, hence the combinatorial structure of the spine is not sufficient to determine its regular neighborhood in the ambient manifolds; what is needed, as shown in the above example, is a kind of Euler number of the normal bundle of the polyhedron. This number, called the gleam, is a color on each region of the polyhedron and turns out to be sufficient to fully reconstruct the regular neighborhood of the polyhedron in the ambient manifold (and hence the whole manifold if it collapses over the polyhedron). A shadow of a 3-manifold is simply a shadow of a 4-manifold whose boundary is the given 3-manifold. Hence, the discussion above shows that it is possible to describe 3-manifolds by means of simple polyhedra whose regions are equipped with numbers. This presentation method will be explained in detail in the subsequent sections.

A natural problem which arises while dealing with shadows is to determine when, given two polyhedra equipped with gleams, they describe the same 3-manifolds. We give a full answer to this question in the present paper, by further developing Turaev's results on this topic and obtaining a calculus for simply connected shadows of 3-manifolds which is strictly analogous to Kirby calculus. More explicitly, we exhibit a set of local modification of polyhedra equipped with gleams which, used in suitable sequences, allow one to connect any two shadows of the same 3-manifold; when one restricts to simply connected shadows, the set of moves needed has a pleasant feature: each move is local, that is it acts only in a contractile subset of the polyhedron, corresponding to a ball in the ambient manifold.

2. Preliminaries

In this section we recall the basic notion of *integer shadowed polyhedron* and the thickening theorem proved by Turaev which allows one to canonically thicken such an object to a 4-manifold. We then give the definition of *shadow of a 4-manifold* and *shadow of a*

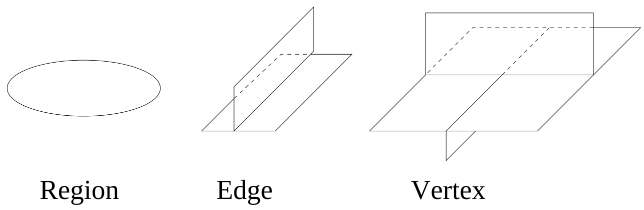

FIGURE 1. The three local models of a simple polyhedron.

3-*manifold*. We also define some modifications, called "moves", which are useful to transform shadows of the same manifold into each other. The main references for this section are Turaev's works [12], [13], and, for an introductory account, [2].

2.1. **Integer shadowed polyhedra.** A *simple* polyhedron is a two dimensional, finite and connected polyhedron which is locally homeomorphic to one o the three models shown in Figure 1. From now on, by the word "polyhedron" we will mean simple polyhedron. Given a polyhedron X, we call the *boundary of* X and denote it as ∂X, the set of points in X which have arbitrarily small neighborhoods homeomorphic to a closed half-plane or to the product of a "T"-shaped trivalent graph with a half open interval, hence ∂X is a trivalent graph; when ∂X is empty, we say that X is closed. We denote by $\mathrm{int}(X)$ the open sub-polyhedron $X - \partial X$ and by $\mathrm{Sing}(X)$ the graph obtained by taking the closure of the set of points not belonging to ∂X were X fails to be a surface.

We will call *regions* the connected components of $X - \mathrm{Sing}(X)$, *vertices of* X the vertices of $\mathrm{Sing}(X)$ of valence exactly four (hence not those corresponding to vertices of ∂X) and *edges* the arcs of the graph $\mathrm{Sing}(X)$. If the closure of a region Y in X contains an arc in ∂X then Y is called a *boundary region*; otherwise it is a *internal region*.

Given a simple polyhedron X, we show now how one can canonically associate an element of $\{0, 1\}$ to each internal region of X.

Let Y be such a region and let $\overline{Y}$ be a compact surface such that Y is homeomorphic to the interior of $\overline{Y}$. The embedding of Y into X extends to a map $i : \overline{Y} \to X$ such that $i(\partial\overline{Y}) \subset \mathrm{Sing}(X)$. This map is not necessarily an homeomorphism, since $i(\partial\overline{Y})$ can pass over the same edge of X more than once. Let P be the open polyhedron which retracts on $\overline{Y}$ and which is constructed so that i extends to a local homeomorphism from P to X. Such a polyhedron can be constructed just by "pulling back" an open neighborhood of $i(\overline{Y})$ in X through the map i. The polyhedron $P - Y$ retracts to a disjoint union of annuli and Möbius strips; then we associate 1 to Y if the number of Möbius strips so obtained is odd and 0 otherwise. We call this number the $\mathbb{Z}_2$-gleam of Y and denote it as $\mathrm{gl}_2(Y)$.

Definition 2.1. An *integer shadowed polyhedron* (X, gl) is a pair of a polyhedron X and a coloring for all the regions of X with colors in the set of half integers, such that, for any internal region Y, the following equation holds: $\mathrm{gl}(Y) - \frac{1}{2}\mathrm{gl}_2(Y) \cong 0 \pmod 1$. If X is a surface the preceding conditions becomes that the gleam be an integer number. The color of a region is called the *gleam*.

Remark 2.2. Any polyhedron can be equipped with gleams in infinitely many different ways so to obtain an integer shadowed polyhedron; indeed, adding any integer to the $\mathbb{Z}_2$-gleam of any region produces a set of gleams which satisfies the above conditions.

2.2. **Polyhedra in 4-manifolds.** In this subsection we investigate how a polyhedron embedded in a 4-manifold can be equipped with the extra structure of integer shadowed polyhedron related to the topology of its regular neighborhood and then we recall Turaev's fundamental thickening theorem. From now on, all the manifolds we will be dealing with will be compact, PL and oriented, unless explicitly stated.

Let X be a polyhedron and suppose that it is properly embedded in a 4-manifold M (that is embedded so that $\partial X \subset \partial M$). Let us be more specific regarding the word "embedded":

Definition 2.3. A polyhedron properly embedded in a 4-manifold is said to be *locally flat* if for each internal point p of X there is a local chart (U, ϕ) of the PL atlas of M such that the image of $X \cap U$ through ϕ is exactly one of the three local pictures of Figure 1 in $\mathbb{R}^3 \subset \mathbb{R}^4$, that is, around each of its points, X is contained in a 3-dimensional slice of M and in this slice it appears as shown in Figure 1.

For the sake of brevity, from now on we will use the word "embedded" for "locally flat properly embedded". The first question we ask ourselves is the following: can we reconstruct the regular neighborhood in a manifold M of a polyhedron X from its combinatorics?

Let for instance, X be homeomorphic to S^2 (probably the easiest polyhedron to visualize). Suppose that X is embedded in an oriented 4-manifold M. It is clear that the answer to our question is "no" since the regular neighborhood of a sphere (and more in general of a surface) is determined by the topology of the surface and by its self-intersection number in the manifold. To state it differently, the regular neighborhood of a surface in a 4-manifold is homeomorphic to the total space of a disc bundle over the surface (its normal bundle) and the Euler number of this bundle is a necessary datum to reconstruct its topology.

Hence we see that, to codify the topology of the regular neighborhood of X in M, we need to decorate X with some additional information; in the case when X is a surface the Euler number of its normal bundle is a sufficient datum. Conversely, the embedding of a surface in a 4-manifold equips the surface with an integer number: the Euler number of is normal bundle.

More in general the following holds:

Proposition 2.4. *Let X be a polyhedron embedded in an oriented 4-manifold M. There is a coloring of the internal regions of X with values in the half integers $\frac{1}{2}\mathbb{Z}$ canonically induced by its embedding in M. This coloring induces a structure of Integer Shadowed Polyhedron on X and hence a gleam on X. Moreover, if ∂X is framed in ∂M then the above coloring can be defined also on the boundary regions of X.*

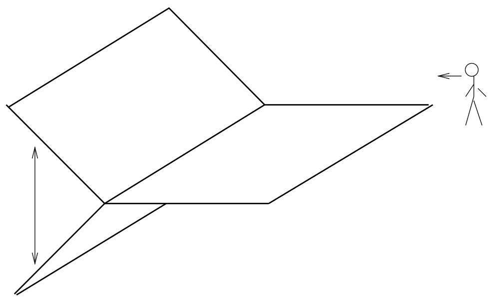

FIGURE 2. The picture sketches the position of the polyhedron in a 3-dimensional slice of the ambient 4-manifold. The direction indicated by the vertical double arrow is the one along which the two regions touching the horizontal one get separated.

Proof. Let D be a region of X and let $\overline{D}$ be the abstract compactification of the (open) surface represented by D. The embedding of D in X extends to a map $i : \overline{D} \to X$ which is injective in $\mathrm{int}(\overline{D})$, locally injective on $\partial\overline{D}$ and which sends $\partial\overline{D}$ into $\mathrm{Sing}(X)$.

For the sake of simplicity, from now on, we suppose that i is an embedding of $\overline{D}$ in X and we omit it from our notation, but the same arguments apply to the general case when $\partial\overline{D}$ runs more than once over an edge of $\mathrm{Sing}(X)$. The regular neighborhood in X of a point p_0 of $\partial\overline{D}$ different from a vertex of X is composed of a half plane (corresponding to $\overline{D}$) glued along a segment to a square in $\mathbb{R}^2$ which is split into two components, corresponding to the other regions touching $\overline{D}$ along its boundary near p_0. Let us consider the three dimensional slice B^3 of M into which a regular neighborhood of p_0 in X is sitting as shown in Figure 1 . Let us fix a transversal direction to $\overline{D}$ in B^3 up to homotopy transverse to $\overline{D}$; this direction coincides with the direction along which the two other regions touching $\partial\overline{D}$ on p_0 get separated (see Figure 2). We can extend the definition of this direction to the whole $\partial\overline{D}$, just by following the boundary of $\overline{D}$ and using the 3-dimensional slices of M containing $\mathrm{Sing}(X)$. That way we obtain a continuous choice of transversal directions to $\overline{D}$ in M along $\partial\overline{D}$. It could happen that after a whole loop around $\partial\overline{D}$ the direction chosen is opposite to the one initially chosen (up to homotopy transverse to $\overline{D}$). In this case, in a local chart near p_0 we add a half-twist to this transverse direction letting it pass through the fourth dimension; there are two possible choices for this half twist according to whether we let the field rotate in a positive or negative direction. More precisely, we fix an auxiliary orientation on $\overline{D}$ and we complete it to an orientation of the transversal 2-planes so that we get the orientation of M; then we let the field rotate in a counterclockwise sense on the 2-plane transversal to $\overline{D}$. Note that the construction does not depend on the orientation chosen for $\overline{D}$ since changing it would contemporaneously change the orientation of the transversal 2-plane.

Of course we could avo this extra half-twist talking about sections a different bundle.

We have described how to obtain a well defined field of transverse directions in M to $\overline{D}$ defined on $\partial\overline{D}$. Now, integrating this field, we can slightly push a copy D' of $\overline{D}$ in M along its boundary so that $\partial D' \cap \partial\overline{D}$ is empty. Now we can put D' in general position with $\overline{D}$, while keeping fixed their boundaries, and count their intersection number g in M. Then, we define the gleam of D to be $g + \frac{1}{2}$ if to construct the above described transverse direction we added an half twist, and g otherwise. It is straightforward to check that the so constructed gleam is integer if and only if the $\mathbb{Z}_2$-gleam of D is zero.

We can analogously perform this construction also in the case when i is not an embedding of $\overline{D}$ in X; that way, we can define a gleam over each region of X.

If X is non closed, we can define the gleam of a boundary region D only when ∂X is framed in ∂M. In this case, the framing over $\partial D \cap \partial X$ defines a section of the normal bundle of D in M over the part of the boundary of D which is contained in the boundary of X, and hence exactly where the above construction could not be applied; on the remaining arcs of ∂D we can still apply the above construction to define a section of the normal bundle of D in M. Hence, we can still construct a section defined over the whole ∂D and hence a gleam, calculated exactly as above. $\boxed{2.4}$

The above proposition shows that an embedding of a simple polyhedron in a 4-manifold, allows one to equip the polyhedron with an Integer Shadowed Polyhedron structure. A partial converse to this fact is given by the following theorem due to Turaev which is fundamental in the theory of shadows:

Theorem 2.5 (Thickening theorem)**.** *Let (X, gl) be a polyhedron equipped with gleams; there exists a canonical reconstruction map associating to (X, gl) a pair (M_X, X) where M_X is a compact and oriented 4-manifold containing an embedded copy of X over which it collapses and such that the gleam of X in M_X coincides with the initial gleam gl. The pair (M_X, X) can be explicitly reconstructed from the combinatorics of X and from gl. Moreover, if X is a polyhedron embedded in an oriented manifold M and gl is the gleam induced on X as explained in the preceding section, then M_X is homeomorphic to the regular neighborhood of X in M.*

We will not recall the proof of the above theorem which, roughly speaking, shows how to thicken "block by block" an integer shadowed polyhedron to obtain its 4-dimensional regular neighborhood. For a detailed account see [12], [13] and, for an introductory one, see [2].

Remark 2.6. When X is standard, i.e. all the regions are discs and $\mathrm{Sing}(X)$ is not a closed curve, then M_X has an handle decomposition induced by X in the sense that each vertex of X corresponds to a 0-handle, each edge to a 1-handle and each region to a 2-handle. In the general case, it is still true that M_X admits a handle decomposition containing no 3 and 4-handles, and such a decomposition can be obtained by a subdivision of X into discs, edges and vertices.

Example 2.7. Let X be a standard spine of an oriented 3-manifold N, then for each region Y of X the $\mathbb{Z}_2$-gleam $\mathrm{gl}_2(Y)$ is zero. So we can construct an integer shadowed polyhedron X from X just by stipulating that the gleam of each region is zero. The application of Theorem 2.5 produces the manifold $N \times [0,1]$.

Remark 2.8. Since the outcome of the thickening procedure are manifolds with an orientation, from now on, all the manifolds we will study will be oriented and all the homeomorphisms will be orientation preserving, unless explicitly stated. It is worth noting that, if (X, gl) is a shadow of a 4-manifold M, then $(X, -\mathrm{gl})$ is a shadow of $\overline{M}$, the manifold obtained from M by changing the orientation.

2.3. **Shadows of** 3 **and** 4**-manifolds.** Having already recalled the thickening theorem, we are ready to give the following:

Definition 2.9 (Shadows of 4 and 3-manifolds)**.** Let M be an oriented, compact 4-manifold, N be a closed 3-manifold, T be a framed trivalent graph inside N and let X be an integer shadowed polyhedron. We say that X is a *strict shadow* or simply a *shadow* of M if M_X is homeomorphic to M, more generally X is said to be a *stable shadow* of M if M is homeomorphic to a 4-manifold obtained from M_X by attaching some three and four handles whose attaching spheres do not intersect $\partial X \subset \partial M_X$.

We say that X is a shadow for the pair (N, T) if ∂M_X is homeomorphic to N by a homeomorphism sending the framed graph ∂X onto T. For instance, when T is empty then X is a *shadow* of N if N is homeomorphic to ∂M_X.

Remark 2.10. Our notation differs somewhat from Turaev's one: what we call "stable shadow" is what Turaev calls shadow of a 4-manifold.

The following is a generalization of the preceding definition to the case of 3-manifolds with boundary but having no spherical boundary components. The reason why we split the definition in two is that we will show the existence of shadows in the general case but then we will prove a calculus for these objects only in the closed case: a proof of the calculus in the case with boundary would get complicated by some technical points.

Definition 2.11. [Shadows of 3-manifolds with boundary.] Let N be a compact 3-manifold whose boundary contains no spherical components, T a framed graph embedded in $\mathrm{int}(N)$ and let X be an integer shadowed polyhedron.

We say that X is a *shadow* for the pair (N, T) if it is possible to split the set of connected components of ∂X into two subsets $\partial_i X$ and $\partial_e X$, respectively called the *internal boundary* and the *external boundary* of X, such that N is homeomorphic to the complement of an open regular neighborhood of $\partial_e X$ in ∂M_X and the homeomorphism sends T to the framed graph $\partial_i X$.

Remark 2.12. The assumption that ∂N does not contain spherical components is technical. Indeed, to construct a shadow of such a manifold, one should allow polyhedra which are not simple but which can also contain arcs with one endpoint contained inside a region and the other endpoint free. For the sake of clarity we exclude these cases and hence, from now on, by 3-manifold we will mean oriented, compact and with (possibly empty) boundary not containing spherical components.

2.4. **The basic moves.** In this subsection we describe a set of moves which allow to produce new shadows from initial ones. Before plunging into the description of these moves some are remarks are in order. In our pictures of integer shadowed polyhedra, we will draw only the local patterns over which each move applies; when a region is entirely contained in one of the drawn patterns we will write on it an half-integer number representing its gleam and, in particular, when no gleam is indicated on such a region then its gleam is zero. Some moves can change the total gleam of a region even if the region is not entirely contained in the local pattern to which the move applies: this is the case of the $1 \to 2$ move, see Figure 5; in these cases, on each region after the application of the move, we will only indicate the total change of its gleam.

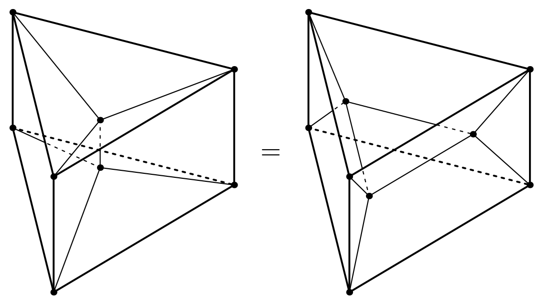

FIGURE 3. The $2 \to 3$ move.

The first move we describe is called the $2 \to 3$-move. Its effect on a integer shadowed polyhedron X is shown in Figure 3.

The second move, called “finger move” or also “lune move” or $0 \to 2$-move, is shown in Figure 4. This move can split (locally it does) the central region of the picture into two regions. To produce an integer shadowed polyhedron one has to choose how to split the gleam of this region into the two gleams of the two new regions; in doing so, one has to ensure that the compatibility equations of Definition 2.1 on the final polyhedron are satisfied.

The third move is the $1 \to 2$-move, drawn in Figure 5. Many remarks are in order here. First of all note that, to draw the picture of the move, we choose an initial embedding of a vertex in $\mathbb{R}^3$ and then draw the position of the boundary curve of a region after the move; the new polyhedron obtained by this move has no embedding in $\mathbb{R}^3$, indeed one region is shifted in a fourth dimension to perform such transformation. The gleams drawn in the picture signify the change in the total gleam of each region; for instance, the total gleam

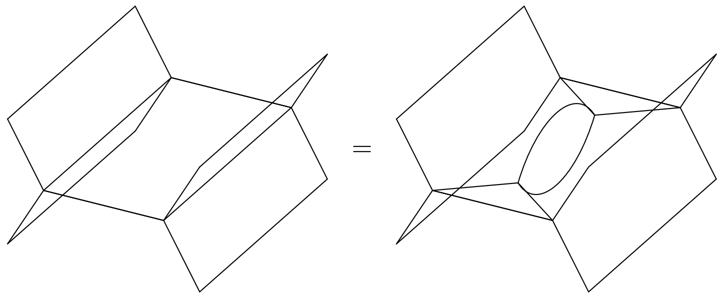

FIGURE 4. The finger move. (TODO: Hidden line removal.)

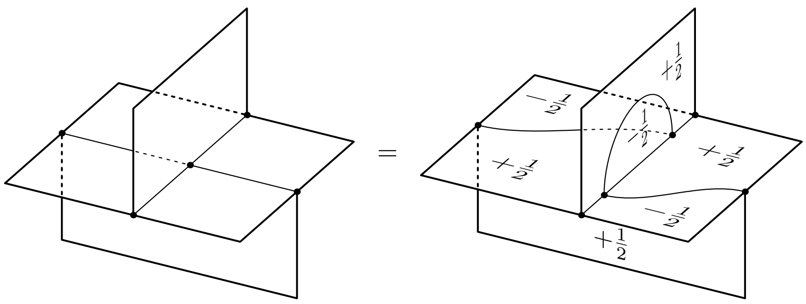

FIGURE 5. The $1 \to 2$ move. On the right hand side, the lower rectangle attaches along the path that runs on the upper rectangle; the result is not embeddable in $\mathbb{R}^3$.

of the integer shadowed polyhedron is changed by the move exactly by $\frac{1}{2}$. Moreover, due to the initial choice of an embedding of the vertex we already mentioned, the picture loses one symmetry, hence, to recover it, we will also consider as a $1 \to 2$-move the specular version of such move. At last, it is possible to see that, using a sequence of finger moves, $2 \to 3$ and $1 \to 2$-moves and their inverses, it is possible to produce the move which has the same picture as the $1 \to 2$-move but which induces a change of gleam on the regions which is exactly the opposite of the one written in the picture.

The fourth move we describe is called ± 1-bubble move and its effect is drawn in Figure 6. Note that this move changes the Euler characteristic of the underlying polyhedron and hence, in particular, of the 4-manifold reconstructed by the integer shadowed polyhedron; on the contrary, we will see later that it does not change the topology of the boundary of that manifold.

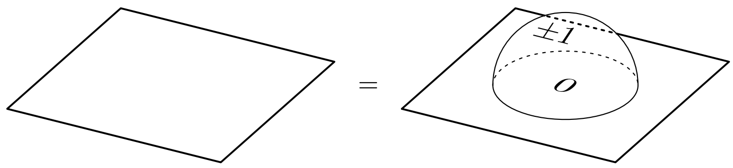

FIGURE 6. The bubble moves.

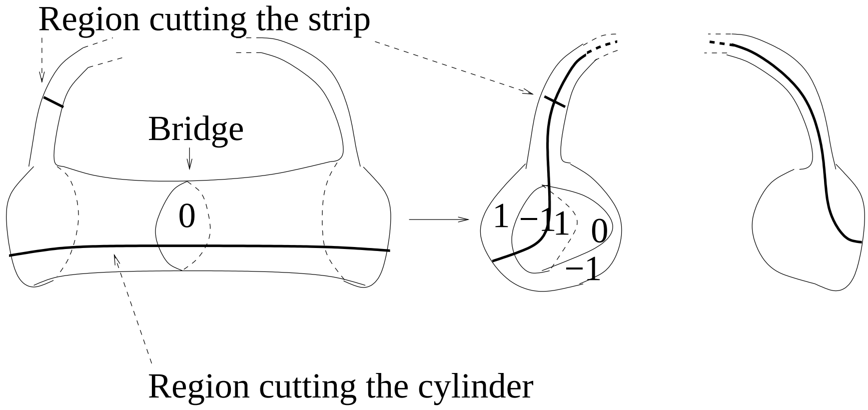

FIGURE 7. In the left part of this figure we show the pattern to which the trading-move applies. It is constituted of a cylinder with a zero gleam disc and of a strip (which could also be twisted) connecting the two ends of the cylinder. Some strands (corresponding to attaching curves of regions of the shadow) could pass over the cylinder or cut the strip. In the right part we show the effect of the trading-move: the cylinder disappears and its two ends are capped off with two zero gleam discs and in one of them a little 0-gleam two disc is attached. The strands which before the move run across the cylinder, are now completed along the strip and linked with the little zero gleam disc as showed in the drawing.

The last move we introduce is called the “trading”-move and it is visible in Figure 7. Unlike the other moves already described, this move does not start from a contractible sub-polyhedron of the initial integer shadowed polyhedron; hence we say that it is “non-local”. One can apply it only when a pattern in the integer shadowed polyhedron is found which coincides with the left part of the figure. The effect of the move dramatically changes the topology of the polyhedron: in particular, the fundamental group changes.

The following theorem, proved by Turaev in [12], resumes the effect of each of the above described moves on the level of the thickening of the integer shadowed polyhedra, and hence answers the question "If X' is obtained from X through the application of a basic move, what is the relation between M_X and $M_{X'}$?".

Theorem 2.13. *Let X and X' be two integer shadowed polyhedra such that X' is obtained from X by applying a move of the following types:*

(1) $2 \to 3$*-move or its inverse;*
(2) *finger-move or its inverse;*
(3) $1 \to 2$*-move or its inverse.*

Then M_X and $M_{X'}$ are homeomorphic by an orientation preserving homeomorphism.

If X' is obtained from X by a ± 1-bubble move then $M_{X'}$ is homeomorphic to the boundary connected sum of M_X and a punctured $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ respectively, hence $\partial M_{X'}$ is homeomorphic (by an orientation preserving homeomorphism) to ∂M_X. Finally, if X' is obtained from X by the application of a trading move, then $M_{X'}$ is obtained from M_X by doing a 4-dimensional surgery along the curve which passes once over the bridge and once over the cylinder of Figure 7. In particular in any of the above cases $\partial M_{X'} = \partial M_X$: the five moves we described do not change the boundary of the thickening of the integer shadowed polyhedra on which they are applied.

3. Shadows of 3-manifolds

Let N be an oriented and compact 3-manifold containing a framed trivalent graph T; in this section we first show how to construct a simply connected shadow X of (N, T) through a theoretical construction and then recall a more actionable recipe invented by Turaev to perform the same task starting from a surgery presentation of (N, T) on a link in S^3. In the second part of the present section we study how two different shadows of a pair (N, T) are related in the case where N is closed. The last paragraph of the section is devoted to study how are the different simply-connected shadows of the same pair related.

3.1. Constructing shadows of a pair (N, T).

In this subsection, starting from a shadow X of N, which by the moment we will suppose to be closed, we describe how to project T on X and construct a shadow of the pair (N, T) or a shadow of the pair $(N', \emptyset)$ where N' is the complement in N of an open regular neighborhood of T. Moreover, when T is a framed knot, we show how to construct a shadow of the manifold N_T obtained from N by doing integer surgery on T. We apply then this construction to produce a shadow of any compact 3-manifold (possibly with boundary) starting from a shadow of S^3.

The restriction of the retraction of M_X onto X gives a projection $\pi : N \to X$; note that M_X is exactly the mapping cylinder of this projection. Moving slightly T by an isotopy in N we can suppose that its projection is a generic trivalent graph in X, i.e. that the mapping cylinder of $\pi : T \to X$ is a simple polyhedron which we will call X_T. By construction X_T is embedded in M_X and $\partial X_T = \partial X \cup T$. In Subsection 2.2 we showed how to give gleams to the so obtained polyhedron X_T embedded in M_X; since by hypothesis T is framed in N it is possible to give gleams to all the regions of X_T.

Then, X_T is a shadow of the pair (N, T); to obtain a shadow of $(N', \emptyset)$ it is sufficient to declare that the boundary components of X_T are external (see Definition 2.11).

In the particular case when T is a framed knot in N, the preceding construction produces an integer shadowed polyhedron X_T whose boundary is one S^1 corresponding to T.

Lemma 3.1. *Let N_T be the 3-manifold obtained from N by doing p-surgery on T. Then the integer shadowed polyhedron X' obtained from X_T by attaching a p-gleam disc to the boundary component corresponding to T is a shadow of N_T.*

Proof. The thickening of a disc attached to X_T is a $D^2 \times D^2$ attached along the solid torus $\partial D^2 \times D^2$ to a regular neighborhood of T in N so that the core $\partial D^2 \times \{0\}$ goes onto T. Moreover, the framing curve $\partial D^2 \times \{1\}$ is identified with a curve which twists p times with respect to the framing curve for T in N. Hence the boundary of the new 4-manifold is obtained from the complement in N of a neighborhood of T by attaching the boundary of the $D^2 \times D^2$ block glued to M_{X_T}; this boundary is the solid torus $D^2 \times \partial D^2$ whose meridian disc is by construction identified with the curve which twists p times around T with respect to its framing. $\boxed{3.1}$

Until now, we started from a shadow X of N and we constructed a shadow of the pair (N, T); in what follows we show how to construct a shadow of any 3-manifold and hence, by the preceding construction of any pair (N, T).

First of all we note that an S^2 with gleam 1 is a shadow of S^3 and we will call it S^2_1. Indeed, such a shadow describes the disc bundle over S^2 with Euler number 1 whose boundary is S^3.

Proposition 3.2. *Given any oriented and compact 3-manifold N possibly with boundary, there exists a simply connected shadow X of N.*

Proof. Each 3-manifold N as in the statement of the proposition admits a presentation as an integer surgery over a framed link contained in the complement of some (possibly knotted) handlebody in S^3. To see this, it is sufficient to "close" the manifold N by attaching to its boundary some handlebodies and then take any surgery presentation in S^3 of the closed manifold obtained that way; up to isotopy, one can suppose that the surgery link does not meet the handlebodies glued to N to close it. Now, inside each of these handlebodies in S^3 draw a framed trivalent graph onto which the corresponding handlebody collapses. That way, one obtains a graph G composed by a link and a set of trivalent graphs (possibly knotted) in S^3. To construct a shadow for the complement in S^3 of an open regular neighborhood of G it is sufficient to consider the shadow projection S_G of G into the shadow S^2_1 of S^3 where all the boundary components of S_G are considered as external. Then a shadow for N is obtained from X_G by filling some of these components (those corresponding to the components of the surgery link) by discs with the appropriate gleams (according to Lemma 3.1) and then to declare the remaining component of ∂X to be external.

Clearly, the shadow one obtains by this procedure is simply connected: indeed it is homotopy equivalent to a bouquet of S^2. 3.2

3.2. **Shadows of links in S^3.** A shadow of a framed link L or, more in general, of a framed trivalent graph in S^3, can be constructed by projecting as explained in the preceding subsection the link L in the shadow S^2_1 of S^3. Although theoretically clear, the above procedure does not shed much light on how to effectively construct a shadow of a framed link in S^3. In [13] Turaev gave an explicit rule to do that starting from the link diagram in $\mathbb{R}^2$ and we now recall it.

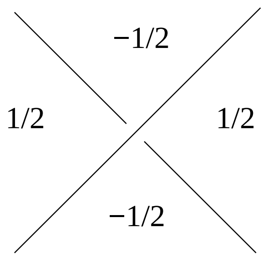

FIGURE 8. How to give gleams to the four regions adjacent to a crossing in a link diagram.

The diagram of L splits $\mathbb{R}^2$ in compact regions R_i and only one non-compact region which we call R_∞. Each crossing of L gives a contribution to the gleam of the four regions incident to the crossing as shown in Figure 8; if we are projecting a trivalent graph instead of a link then the projection of the vertices will be some trivalent vertices in the projection which will give no contribution to the gleams of the adjacent regions. The total gleam of a region is the sum of all these contributions; the total gleam of R_∞ will be 1 plus the sum of all these contributions. This produces a subdivision of S^2 in regions equipped with gleams which sum up to 1. To give gleams to the annular regions corresponding to the cylinders over the components of L in the mapping cylinder of the projection of L in S^2_1, one assigns to each such annulus a gleam equal to the framing of the corresponding component of L with respect to the framing induced by the projection in $\mathbb{R}^2$. Analogously, in the case of trivalent graphs, to give gleams to the rectangular regions attached to the edges of the graph, one counts the number of half twists the framing of the edge does over the edge with respect to the framing induced by the projection on S^2_1 and then assigns this number divided by two to the rectangular region of the mapping cylinder corresponding to the edge.

It is worth note that there are other possible ways to construct shadows of links in S^3 and that the preceding rule always gives as a result a polyhedron retracting on S^2. In the following exercise, we show an example of contractible shadow of the Hopf link.

Exercise 3.3. Let H be the integer shadowed polyhedron obtained by attaching a 1-gleam disc on the core curve of an annulus; show that this is a shadow of the Hopf link.

In the already cited paper, Turaev proved a result which we can restate as follows:

Theorem 3.4. *If two link diagrams are related by a combination of Reidemeister moves then the shadows of the links constructed as explained above are connected by a sequence of* $1 \to 2$, $2 \to 3$, *finger-moves and their inverses. In particular, any isotopy of a link can be translated in the shadow world in terms of the first three moves we described in Subsection 2.4.*

3.3. **The calculus for shadows of 3-manifolds.** Having already shown that any compact 3-manifold admits a shadow, we now address the question of how are any two shadows of the same 3-manifold related.

We have already seen in Theorem 2.13 that if two integer shadowed polyhedra are related by a sequence of moves of the five types we introduced in Subsection 2.4, then they are shadows of the same 3-manifold. The problem now is to prove the converse.

To do that, here, we will limit ourselves to the closed case, but we stress that the more general result in the non-empty boundary setting can be proven by means of similar techniques and dealing with a series of minor technicalities which we prefer to skip now for the sake of simplicity. Hence we prove here the following:

Theorem 3.5. *[Calculus for shadows of 3-manifolds] Let N be a closed 3-manifold, T be a framed graph embedded in N and let X_1 and X_2 be two shadows of (N,T). Then the two shadows can be connected by a sequence of moves of the following types:*

(1) $2 \to 3$-*move and its inverse;*
(2) *finger-move and its inverse;*
(3) $1 \to 2$-*move and its inverse;*
(4) ± 1-*bubble move and its inverse;*
(5) *trading-move and its inverse.*

Proof. Applying, if necessary, some 1-bubble-moves, we can suppose that the singular sets of both X_1 and X_2 are non-empty. Moreover, applying finger-moves, we can suppose that they are connected graphs and that all the regions of X_1 and X_2 are discs. We split the rest of the proof in three steps:

Step 1: finding an "almost standard" form for X_1 and X_2.

The application of a 1-bubble move on X_1 creates a little sphere S with total gleam 1 composed of two discs. Using $2 \to 3$ and finger-moves we want to slide this sphere "inside" the singular set of X_1. More precisely, let Y be the region over which we applied the bubble-move and let e be an edge of X_1 contained in the closure of Y in X. Let moreover

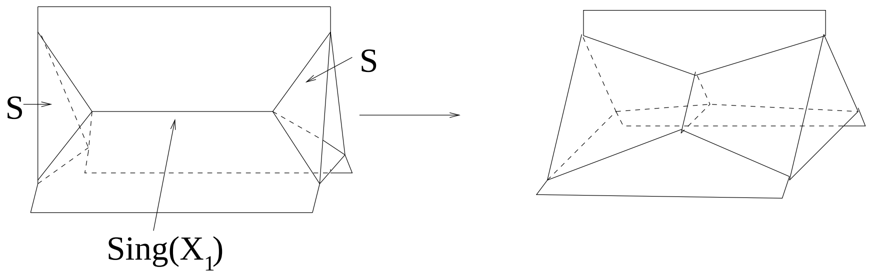

FIGURE 9. In the left part of this figure we show an edge of $Sing(X_1)$ non belonging to $Tree(X_1)$. At each of its ends, a sub disc of the sphere S is visible. In the right part we show how to apply a $2 \to 3$-move to obtain a cylinder with a zero gleam disc and three strands (corresponding to the 3 regions touching the edge) running across it.

Y' be a region of X_1 adjacent to Y along e (it could happen that $Y' = Y$ but this does not cause problems to our proof since locally, near e the regions Y and Y' appear as different regions). Using a finger-move we can slide Y' over a subdisc of the 1-gleam disc contained in the sphere S; the result is X_1 with a "bubble" at the center of e.

Now let Tree(X_1) be a maximal subtree of Sing(X_1) containing the edge e. Starting from e and using $2 \to 3$ and finger-moves we can slide S inside the whole Tree(X_1), so that, at the end of this operation, we are left with a new shadow of (N, T) as follows:

- the vertices of X_1 have been replaced by 4-punctured spheres with a set of six non-intersecting curves on it representing the attaching curves of the regions around the vertex;
- the edges of Sing(X_1) belonging to Tree(X_1) have been substituted by cylinders with three parallel curves on it corresponding to the three attaching curves of the region around the edges;
- the edges of Sing(X_1) non-belonging to Tree(X_1) survive in the new shadow and the structure of the polyhedron around each of them is identical to the one shown in the left part of Figure 9;

After having obtained this new integer shadowed polyhedron, we further transform it by means of $2 \to 3$-moves near each of the edges of Sing(X_1) $-$ Tree(X_1) as shown in Figure 9 to get a cylinder cut by a zero gleam disc over which three attaching curves pass.

At the end of this sequence of moves, we are left with a shadow X_1' of (N, T) constructed as follows: on a 1-gleam sphere, $2n$ zero gleam discs are identified (not necessarily by an orientation reversing homeomorphism) in pairs (where n is the number of edges in Sing(X_1) $-$ Tree(X_1)) so that we get a genus n surface with n zero-gleam meridian discs. Some regions (corresponding to the original regions of X_1) are attached to this surface along embedded and non intersecting curves; moreover, a copy of $T \times [0,1]$ is attached to this surface along an embedding of $T \times \{0\}$.

The vertices of X_1' correspond to the intersections of the attaching curves of the regions and of $T \times [0,1]$ with the zero-gleam discs; moreover, each such disc intersects exactly three of this curves.

We repeat the above procedure for X_2 to get a new shadow X_2'.

Step 2: Reducing X_1' and X_2' to surgeries on S_3. In this step we apply n trading-moves to X_1' and get a new shadow X_1'' whose form is the following: a sphere with total gleam 1 containing a set of attaching curves (no longer embedded) of the regions corresponding to the regions of X_1 and to the graph T; moreover, n new zero-gleam disc regions are attached to the sphere as in the right part of Figure 7, such regions correspond to the 2-handles traded for the 1-handles of the genus g surface of the preceding Step. That way, one obtains a shadow projection of a graph in S^3, with some surgeries on some S^1-components corresponding to the boundaries of the regions of X_1 and to the n zero-framed meridians added during the trading move. Apply the same procedure to X_2' to get X_2''.

Step 3: Using the Kirby calculus. The shadow X_1'' is a shadow of (N,T) of the same form as the one obtained by projecting into a S_1^2 (the shadow of S^3) a framed graph (corresponding to T) and a link L contained in S^3 (see Subsection 3.2) and then attaching discs to the boundary components corresponding to the link (see Lemma 3.1); the same holds for X_2''. Now that we modified X_1 and X_2 into these particular forms X_1'' and X_2'', we apply the Kirby calculus to connect X_1'' to X_2''.

It is known that any two surgery presentations on links in S^3 of the same closed 3-manifold are connected by a sequence of the following moves:

(1) isotopy;
(2) adding or deleting an unknot with framing ± 1 separated from the surgery link;
(3) band sum of two components of the surgery link.

If one takes into account also the trivalent graph T embedded in N, one should also use some band sum of an edge of the graph with some components of the surgery link: this last move corresponds to moving T inside N letting an edge pass through the meridian disc of a component of the surgery link.

It is clear that these moves are sufficient to get from any surgery presentation in S^3 of (N,T) to any other one.

In Subsection 3.2 (see Theorem 3.4) we already mentioned the fact that any isotopy on a link can be translated, in terms of the shadow projection of the link on the shadow S_1^2 of S^3, into a set of $1 \to 2$, $2 \to 3$, finger-moves and their inverses. This means that if we modify by an isotopy the surgery link L or the framed graph T associated to X_1'', we only modify X_1'' by some of the moves we listed in the statement of the theorem.

Regarding the move of adding an unknot with framing ± 1, note that, in terms of shadow moves, X_1'' gets modified by attaching a ± 1-gleam disc to a little closed curve inside a region (see Lemma 3.1 and Subsection 3.2). This is a ± 1-bubble move, which we allow in our calculus.

We are left to deal with the band sum operation. First of all, up to isotopy of L, which we showed we are allowed to do, we can suppose that the framing of the link is the diagrammatic one. This means that to get N from S^3 it is sufficient to do 0-framed surgery with respect to the blackboard framing of the link diagram of L. Translated in terms of the shadow X_1'', this means that we can modify it by a sequence of $1 \to 2$, $2 \to 3$, finger-moves and their inverses to get a shadow obtained as follows: project the link L and the framed graph T in S_1^2 by using the diagram in $\mathbb{R}^2$ of $L \cup T$ as explained in Subsection 3.2, then attach zero gleam discs to the boundary components corresponding to L.

Now, doing the band sum operation, corresponds to letting a sub arc of the attaching curve of a disc D_i attached to a component of l_i of L, slide over another disc D_j attached to another component l_j of L. We can achieve this sliding using only finger-moves, $2 \to 3$-moves and their inverses. Indeed, to begin the slide one uses a finger-move which shifts an arc in the boundary of D_i over a zero gleam sub-disc D_j' of D_j. Then D_j is split by this arc in two sub discs D_j' and D_j'' both having zero gleam. One then continues sliding the arc over D_j'' using finger moves and if necessary (i.e. when the sliding arc passes over a vertex in $\partial D_j''$), some $2 \to 3$-moves. It is clear that the result is the shadow of the band sum of the two components of the link: indeed we have exactly slid the two handle corresponding to D_i in the 4-manifold $M_{X_1''}$ over the two handle corresponding to D_j.

To conclude, X_1'' and X_2'' are connected by the sequence of moves which translates the sequence of Kirby-calculus moves and hence X_1 and X_2 are equivalent under the equivalence relation generated by the moves listed in the statement of the theorem. [3.5]

3.4. **The simply-connected case.** In this subsection we prove the calculus for simply connected shadows of a pair (N, T).

Theorem 3.6 (Calculus for simply connected shadows.)**.** *Let N be a closed 3-manifold, T be a framed graph embedded in N and let X_1 and X_2 be two simply connected shadows of (N, T). Then the two shadows are connected by a sequence of moves of the following types:*

(1) $2 \to 3$-*move and its inverse;*
(2) *finger-move and its inverse;*
(3) $1 \to 2$-*move and its inverse;*
(4) ± 1-*bubble move and its inverse;*

Remark 3.7. It is worth note that in the simply connected case all the moves needed to get a calculus for 3-manifolds have local nature.

Proof. To prove the theorem it is sufficient to show that in the proof of Theorem 3.5 one can avoid using trading-moves. Note that, in that proof, Step 1 transforms the shadow X_1 into X_1' which is simply-connected if and only if X_1 is, since only $2 \to 3$ and finger-moves are used in this Step. Moreover, trading-moves are used only during Step 2 to transform the shadow X_1' into X_1''. Here we show how to achieve the result of Step 2 in the simply connected case, without using trading-moves.

First of all, by using ± 1-bubble moves and $1 \to 2$, $2 \to 3$, finger-moves and their inverses, it is possible to add to the shadow X_1' two discs R and R_0 attached along two trivial curves c and m such that m is link around c as shown in Figure 10, obtaining another simply connected shadow of (N, T) which we will still call X_1'. This modification corresponds in the Kirby calculus world to the sequence of moves (band sums and additions/deletions of ± 1-framed unknots) adding to a surgery presentation of a 3-manifold a zero framed unknotted curve together with its zero framed meridian; we can perform this sequence on a link contained in a little ball in ∂M_{X_1}. Since, as already noticed, adding a ± 1-framed unknot to such a link corresponds to performing ± 1-bubble move on the shadow and doing band sums corresponds to sliding the bubbles over each other, such a sequence can be easily translated in terms of shadow moves different from the trading-moves.

Let now c' be a curve passing exactly once on a bridge to which, following the proof of Theorem 3.5, we would apply in Step 2 a trading-move. Since X_1' is simply-connected then the curve c is homotopic in the underlying polyhedron to c'. It is a standard fact that the homotopy between $c = \partial R$ and c' can be substituted by an homotopy which can be decomposed into little steps corresponding to $1 \to 2$, $2 \to 3$, finger-moves an their inverses at the level of the underlying polyhedra. More explicitly, the polyhedron $X_1 \cup R$ and the polyhedron $X_1 \cup R'$ where R' is a disc glued to X_1 along c' are connected by a sequence of naked moves. Unfortunately, in general such a sequence cannot be lifted to a sequence of integer shadowed moves.

We want to show now that, using the particular pattern around R and R_0, it is possible to modify this homotopy so that the moves into which it is decomposed are shadow moves and hence take into account the gleams of the regions over which they act.

First of all, any $1 \to 2$, $2 \to 3$ and finger-move at the level of the polyhedron underlying X_1' can be performed at the level of the integer shadowed polyhedron. Problems can surge when dealing with the inverses of these moves: for instance, when an inverse finger-move is applied, the disappearing region has to be equipped with zero gleam. In each of these problematic situations a sub arc α of ∂R slides over a disc region A with gleam g; moreover, the number of vertices in ∂A can be 2 or 3, the second case corresponding to when A is the disappearing region of an inverse $2 \to 3$-move.

We leave it as an exercise to check that it is possible to freely slide the meridian disc R_0 along ∂R by using only $2 \to 3$-moves and their inverses (see [13]), so, we can push the region R_0 and bring it over the sub arc α of ∂R (see Figure 10).

We claim now that, through a sequence of $1 \to 2$, $2 \to 3$, finger-moves and their inverses, it is possible to use the disc R_0 to shift a unit of gleam from the region A to the other region B adjacent to α or viceversa. More precisely, there is a sequence of shadow moves acting only in a neighborhood of $R \cup R_0 \cup A \cup B \cup R_- \cup R_+$ (see Figure 10) whose final effect is to leave unchanged the polyhedron underlying X_1' and to change only the gleams of A and B and R respectively by 1, -1 and ± 1.

Before proving the existence of this sequence, we show how to conclude once such a sequence is known.

Using the sequence (or its inverse version) a suitable number of times, we reduce ourselves to the cases when A has zero gleam and hence the inverse of the finger-move or of the $2 \to 3$-move is performable or to the case where A has gleam $\frac{1}{2}$ where the inverse of the $1 \to 2$-move is performable. This way, we showed that the homotopy bringing the curve $c = \partial R$ onto the curve c' can be translated in terms of shadow moves by means of the particular pattern near R_0 and a suitable (not yet exhibited) sequence. The result is the polyhedron obtained from X_1' by attaching a disc (R) along c' and a zero gleam disc (R_0) on the boundary of a little circle in X_1' split by c' in two subdiscs having gleams respectively equal to 1 and -1 (a copy of the local pattern shown in Figure 10). It can be checked that, by a applying a sequence of $1 \to 2$-moves and their inverses to R having the effect of sliding part of ∂R over the core of the bridge over which c' passes, we can change the gleam of R by any integer amount, in particular bringing it to 0 or $\frac{1}{2}$. At last, if the gleam of R is $\frac{1}{2}$ then, we can apply a $1 \to 2$-move to R along a subarc of its boundary to change it to 0.

Now, we are left with a 0-gleam disc (R) whose boundary passes exactly once over a cylinder as in the left part of Figure 7; it is a standard exercise to see that, by means of $2 \to 3$, finger-moves and their inverses, we can destroy the cylinder together with R obtaining exactly the pattern shown in the right part of Figure 7.

This proves that, in the simply connected case, one can substitute all the trading moves by a sequence of finger-moves, $1 \to 2$, $2 \to 3$-moves and their inverses.

We are now left to prove the existence of the sequence of moves around $R_0 \cup A \cup B \cup R_- \cup R_+ \cup R$ having the effect of shifting a unit of gleam from A to B. Instead of showing this directly by exhibiting the sequence which is somehow complicated, we prefer here another approach which uses the strict relations existing between shadows and framed trivalent graphs.

Consider the two little discs R_+ and R_- bounded by the union of α and ∂R_0 having gleams respectively 1 and -1 (see Figure 10). The sphere S formed by R_0, R_+ and R_- has zero total gleam and so its regular neighborhood in M'_{X_1} is the total space of a trivial disc bundle over S. Then, the integer shadowed polyhedron with boundary represented by a small regular neighborhood of S in X_1 is the shadow of a framed trivalent graph G (whose edges correspond to the union of the intersections of the boundary of the neighborhood of S in X_1 with A, B and R) contained in $\partial M_S = S^2 \times S^1$. We will now show that this framed trivalent graph in $S^2 \times S^1$ can be isotoped so that the edges corresponding to the regions A and B get twisted respectively by $+1$ and -1; this isotopy can be translated in terms of shadow moves on the integer shadowed polyhedron S and hence it produces exactly the sequence we were searching for.

To better understand how is the framed graph G positioned in $S^2 \times S^1$ we first study another simpler graph G', very similar to this one. Let G' be the graph whose shadow

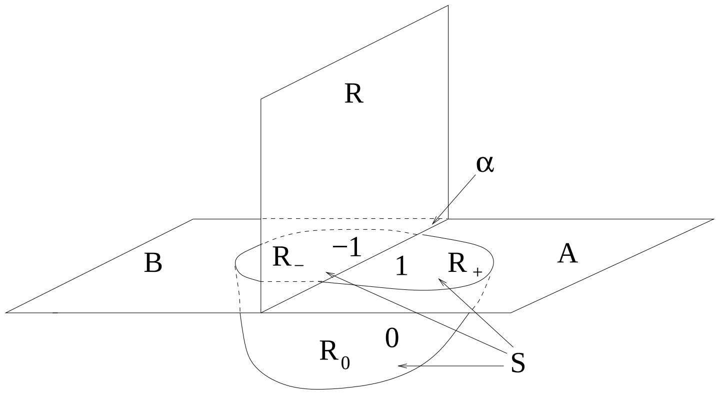

FIGURE 10. The local pattern near α.

projection in a sphere with zero gleam is a theta curve splitting the sphere in three zero gleam regions. Recalling the thickening procedure, we see that the shadow projection of G' can be viewed as a spine of a 3-manifold (in this case $S^2 \times [0,1]$) equipped with zero gleams: hence as a shadow of the manifold $S^2 \times D^2$. Moreover the framed graph G' lies in a S^2 slice of $S^2 \times S^1$: the boundary of the thickening of the shadow which coincides with the double of $S^2 \times [0,1]$. So, to summarize what described until now, we exhibited two different shadows with boundary whose underlying polyhedra coincide and whose thickening are both equal to $S^2 \times D^2$ but whose boundaries form two different framed graphs G and G' in $\partial(S^2 \times D^2) = S^2 \times S^1$. Moreover we already saw that G' is contained in a S^2 slice of $S^2 \times S^1$.

Let us now clarify the position of G' in $S^2 \times S^1$. The difference between G and G' is concentrated only in the position in $S^2 \times S^1$ of one edge of the theta graph. More precisely, G splits the sphere S in three regions one of which has zero gleam and the other two having gleams respectively 1 (R_+) and -1 (R_-), while G' splits S in three 0 gleam discs. This means that, in the manifold M_S which is a trivial D^2 bundle over S^2, if we orient arbitrarily R_+ and look at the section of its unit normal bundle described on its boundary by the two edges of the graph whose projection form ∂R_+, then this section twists exactly once with respect to the trivial section (the one which extends over R_+). Analogously, the obstruction to extend the section of the $S^1 = \partial D^2$-bundle described on ∂R_- by the two edges of G corresponding to the boundary of R_- is -1 (see Section 2.2).

This means that the graph G, exactly as G', is almost completely contained in a S^2-slice of $S^2 \times S^1$ except for a sub arc of the edge of G whose projection separates R_+ form R_- which describes a whole loop around the essential S^1 of $S^2 \times S^1$.

Summarizing the above observations, in Figure 11 we show the graph G in the Kirby presentation of $S^2 \times S^1$ given by zero surgery on the unknot. In the same figure we show

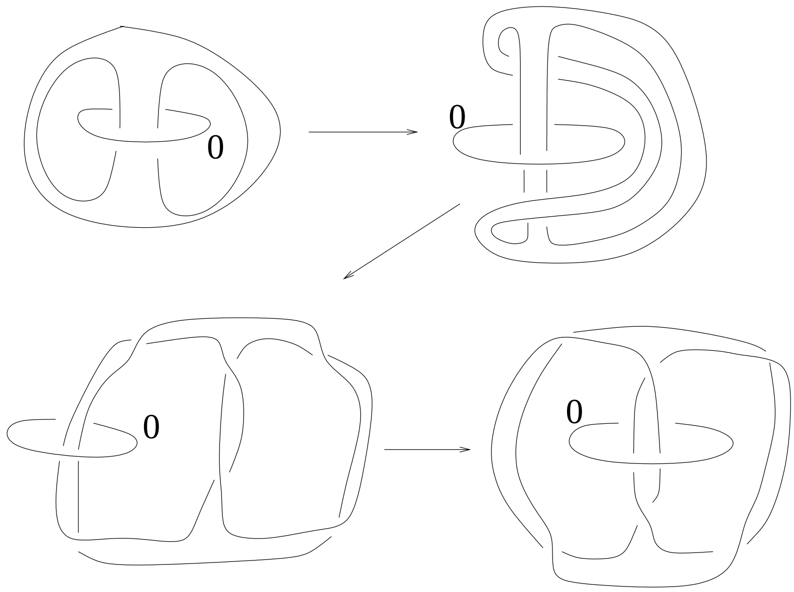

FIGURE 11. In this figure we show the isotopy of the framed graph G in $S^2 \times S^1$ whose effect is to add a twist to one edge of the graph and subtracting one to the other two. The first move is a sliding of the left edge of the graph over the 0-framed unknot linked with the central edge; all the other moves are isotopies.

an isotopy of G changing the framing of the two edges of G corresponding to A and B respectively by 1 and -1 full twist. Then this isotopy can be translated in terms of shadow moves of the integer shadowed polyhedron S and the effect of this sequence (which, of course, can be seen as sequence of moves on X_1' acting only near S) is shifting an amount of gleam from the region A to the region B equal to -1 and changing the gleam of R by ± 1. 3.7

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